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WEIGHTED COMPOSITION OPERATORS ON SEQUENCE SPACES OF ENTIRE FUNCTIONS

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Abstract: In this paper we characterize the weighted composition operators on sequence spaces of entire functions and also make an effort to characterize compactness, closed range and invertibility of these operators.

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1. Introduction and Preliminaries

Let (X, T) be a Hausdorff locally convex space and $U(x)$ be the fundamental system of balanced, convex and absorbing neighbourhoods of origin. For $u \in U(x)$, set $p_u = \sup\{|f(x)| : f \in u^0\}$, where u^0 is the polar of u. Then p_u is a seminorm. Moreover, $p_u = \inf\{\alpha > 0 : x \in \alpha u\}$. Let $D = \{p_u : u \in U(x)\}$. Then D is the family of continuous seminorms generating the topology T on X. We denote by $E(X)$, the class of all sequences $x = \{x_n\} \in X$ such that $\{p_u(x_n)\}^{\frac{1}{n}}$ tends to zero as n tends to infinity for each $u \in U(x)$. For each $x = \{x_n\} \subset E(X)$ we define $P_u(x) = \sup(p_u(x_n))^{\frac{1}{n}}$. Then P_u satisfies the following properties:

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1. $P_u(x) \geq 0$,
2. $P_u(x+y) \leq P_u(x) + P_u(y)$,
3. $P_u(\alpha x) \leq A(\alpha) P_u(x)$, $x, y \in E(X)$,

where α is a scalar and $A(\alpha) = \max\{|\alpha|, 1\}$. Srivastava [6] proved in his paper that P_u is a paranorm on $E(X)$ and $(E(X), P_u)$ is a paranormed space for each $u \in U(x)$. If we consider VP_u the superemum of the topologies induced by all paranorms $P_u, u \in U(x)$, then $(E(X), VP_u)$ is a topological vector space. It is proved in [6] that if X is complete, then $(E(X), VP_u)$ is also complete.

If the mapping $u : N \to C$ and $v : N \to N$ are such that $u.f \circ v \in E(X)$, then a weighted composition transformation $m_{u,v} : E(X) \to E(X)$ is defined as $m_{u,v} f = u.f \circ v$ for every $f \in E(X)$. In case $m_{u,v}$ happens to be continuous, we name it as a weighted composition operator.

A linear transformation A defined on a locally convex space (X, T_P) where P is the family of seminorms on X is an isometry if for every $p \in P$, $p(Af) = p(f)$, for every $f \in X$. For more details about sequence spaces one can refer to ([6],[7]) and references therein, whereas the study of weighted composition operators on some function spaces are considered by ([1],[2],[3],[4],[5]) etc. By $B(E(X))$ we denote the set of all bounded linear operator from $E(X)$ into itself.

In this paper we plan to study the weighted composition operators on sequence spaces of entire functions $E(X)$.

2. Main Results

Theorem 2.1. *Let $m_{u,v} : E(X) \to E(X)$ be a linear transformation. Then $m_{u,v}$ is continuous if and only if for each $\epsilon > 0$ there exists $b > 0$, $M > 0$ such that*

1. *$\{n : \frac{n}{v(n)} > b\} \cap \{n : u(n) \geq \epsilon\}$ is a finite set,*
2. *$|u(n)|^{\frac{1}{n}} \leq M$ for every $n \in N$.*

Proof. Assume first that $m_{u,v}$ is continuous. If the condition (i) were false, there exist $\epsilon > 0$ such that the set $\{n : \frac{n}{v(n)} > k\} \cap \{n : u(n) \geq \epsilon\}$ is an infinite for each $k \in N$. Let $\{n_k\}$ be an infinite sequence of natural numbers such that $\frac{n_k}{v(n_k)} \geq k$ and $u(n_k) \geq \epsilon$. Suppose $u \in U(X)$ and $0 \neq y \in u$. Take $x = \frac{y}{p_u(y)}$. Then $p_u(x) = 1$. Let $f_{n_k} : N \to X$ be the function defined by $f_{n_k} = R^{n_k} x \chi_{v(n_k)}$

where $0 < R < 1$. Now,

$$P_u(f_{n_k}) \;=\; (p_u(R^{n_k}x))^{\frac{1}{v(n_k)}} = R^{\frac{n_k}{v(n_k)}}.(p_u(x))^{\frac{1}{v(n_k)}} \leq R^k \to 0 \text{ as } k \to \infty.$$

Let $x \in X$ be such that $x \in u$ for some $u \in U(X)$. So

$$\begin{aligned} P_u(m_{u,v}f_{n_k}) &= \sup\{(p_u(u(m)f_{n_k}(v(m))))^{\frac{1}{m}}, m \geq 1\} \\ &\geq (p_u(u(n_k)R^{n_k}x))^{\frac{1}{n_k}} \\ &= \epsilon^{\frac{1}{n_k}}.R(p_u(x))^{\frac{1}{n_k}} \\ &\geq \epsilon R, \end{aligned}$$

this shows that $m_{u,v}f_{n_k} \not\to 0$, which is a contradiction. Hence the condition (i) must be true. Next if the condition (ii) were false, then for every $k \in N$, there exists $n_k \in N$ such that $u(n_k)^{\frac{1}{n_k}} > k$. Let $u \in U(X)$ be such that $x \in u$ and $p_u(x) = 1$. Let $f_{n_k} = \chi_{v(n_k)}.\frac{x}{u(n_k)}$. Then

$$\begin{aligned} P_u(f_{n_k}) &= \Big[p_u(\frac{x}{u(n_k)})\Big]^{\frac{1}{v(n_k)}} \\ &< (\frac{1}{k})^{\frac{n_k}{v(n_k)}} \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

But $P_u(m_{u,v}f_{v(n_k)}) \geq (p_u(u(n_k).f_{v(n_k)}(v(n_k))))^{\frac{1}{n_k}} = (p_u(x))^{\frac{1}{n_k}}$ which $\not\longrightarrow 0$ as $k \longrightarrow \infty$, which is again a contradiction. Hence the condition (ii)must be true.

Conversely, suppose that the condition (i) and (ii) are valid. We first prove that $m_{u,v}$ is an into map. Let $\epsilon > 0$ be given. Then for $f \in E(X)$ and $\epsilon > 0$, we have $(p_u(f(v_n)))^{\frac{1}{v(n)}} < \epsilon$ for all except finitely many values of n. Now

$$\begin{aligned} (p_u(u(n)f(v(n))))^{\frac{1}{n}} &= |u(n)|^{\frac{1}{n}}(p_u(f(v_n))^{\frac{1}{v_n}})^{\frac{v(n)}{n}} \\ &\leq M\epsilon^{\frac{1}{b}} \end{aligned}$$

for all but finitely many values of n. Hence $u.f \circ v \in E(X)$. The continuity of $m_{u,v}$ follows from the identity

$$\begin{aligned} p_u(m_{u,v}f) &= \sup\{(p_u(u(n)f(v(n)))^{\frac{1}{n}} : n \geq 1\} \\ &\leq MP_u(f \circ v) \\ &\leq M\delta P_u(f) \end{aligned}$$

for some $\delta > 0$. Hence $m_{u,v}$ is continuous. $\square$

Theorem 2.2. *Let $m_{u,v} \in B(E(x))$. Then $m_{u,v}$ is invertible if and only if*

1. *v is invertible,*
2. *$\{n : \frac{n}{v^{-1}(n)} \geq b\} \cap \{n : \frac{1}{uov^{-1}(n)} \geq \epsilon\}$ is a finite set,*
3. *$|\frac{1}{uov^{-1}(n)}|^{\frac{1}{n}} \leq M$ for every n.*

Proof. We first assume that $m_{u,v}$ is invertible with its continuous inverse G. For every $n \in N$, $(m_{u,v} \circ G)(e_n) = e_n = (G \circ m_{u,v})(e_n)$, so that $u.(Ge_n) \circ v = e_n$, which implies that $u(n) \neq 0$. If v is not surjection, then $m_{u,v}e_n = 0$ for every $n \in N \backslash v(N)$ so that $m_{u,v}$ has non-trivial kernel, which contradict the injectivity of $m_{u,v}$. Hence v must be surjective. Next if v is not an injection, then for two distinct positive integers n_1 and n_2, $v(n_1) = v(n_2) = n_0$ (say). Now either $u(n_1) = u(n_2)$ or $u(n_1) \neq u(n_2)$. In case $u(n_1) = u(n_2)$, then we have

$$\begin{aligned} (m_{u,v}f)(n_1) &= u(n_1).f(v(n_1)) \\ &= u(n_2).f(v(n_2)) \\ &= (m_{u,v}f)(n_2). \end{aligned}$$

Thus $g \in E(X)$ with $g(n_1) \neq g(n_2)$ will not be in the range of $m_{u,v}$. Hence v must be injective. Further, in case $u(n_1) \neq u(n_2)$, we have $\chi_{(n_1,n_2)}$ is not in the range of $m_{u,v}$. Hence v must be invertible. Set $q = v^{-1}$ and $p : N \to C$ be defined as $p(n) = \frac{1}{u(q(n))}$. Then $G = m_{p,q}$. By continuity of G, $\{n : \frac{n}{q(n)} > b\} \cap \{n : p(n) > \epsilon\}$ is a finite set and $|\frac{1}{uov^{-1}(n)}|^{\frac{1}{n}} \leq M$ for every $n \in N$.

Conversely, assume that the conditions of the theorem are true. Define the mapping $p : N \to C$ and $q : N \to N$ as $p(n) = \frac{1}{u(v^{-1}(n))}$ and $q(n) = v^{-1}(n)$ respectively for all $n \in N$. Then in view of Theorem 2.1, $m_{p,q}$ is the continuous inverse of $m_{u,v}$. $\square$

Theorem 2.3. *Let $m_{u,v} \in B(E(X))$. Then $m_{u,v}$ has closed range if and only if u^r is bounded away from zero on $[Z(u)]'$, where $u^r(n) = |u(n)|^{\frac{1}{n}}$.*

Proof. Suppose first that the condition of the theorem is true. We prove that $m_{u,v}$ has closed range. Let $h \in E(X)$ be such that $m_{u,v}h_n \to h$ for some sequence $\{h_n\}$ in $E(X)$. Then for $\epsilon > 0$, $0 < \epsilon < 1$, we have $P_u(m_{u,v}h_n - m_{u,v}h_m) < \epsilon$ for all $n, m > n_0$. This implies that

$$\begin{aligned} \delta \sup_{[Z(u)]'} (p_u(h_n(v(k)) - h_m(v(k)))^{\frac{1}{k}} &\leq \sup(|u(k)|^{\frac{1}{k}} p_u(h_n(v(k))h_m(v(k)))^{\frac{1}{k}} \\ &\leq P_u(m_{u,v}h_n - m_{u,v}h_m) \\ &< \epsilon, \text{ for all } n, m \geq n_0. \end{aligned}$$

or $[p_u(h_n(v(k)) - h_m(v(k)))]^{\frac{1}{k}} < \frac{\epsilon}{\delta}$. Therefore $[p_u(h_n(v(k)) - h_m(v(k)))]^{\frac{1}{v(k)}} \leq (\frac{\epsilon}{\delta})^{\frac{k}{v(v(k))}} < (\frac{\epsilon}{\delta})^b$, for all $n, m \geq n_0$ and for all $k \in [Z(u)]^{'}$, where $b = \inf\{\frac{k}{v(k)} : k \in [Z(u)]^{'}\}$, for each $n \in N$, define

$$g_n(k) = \begin{cases} h_n(k), \text{ if } k \in [Z(u)]^{'} \bigcap T(N); \\ 0, \text{ elsewhere.} \end{cases}$$

Then $g_n \in E(X)$ for each $n \in N$ and $P_u(g_n - g_m) = \sup\{p_u(h_n(v(k)) - h_m(v(k))) : k \in [Z(u)]^{'}\}^{\frac{1}{k}} < \epsilon$ for all $n, m \geq n_0$. This shows that $\{g_n\}$ is a cauchy sequence in $E(X)$. Therefore there exists $g \in E(X)$ such that $g_n \to g$. By continuity of $m_{u,v}$, $m_{u,v}g_n \to m_{u,v}g$. Hence $m_{u,v}g = h$, so that $h \in \text{ran } m_{u,v}$ which proves that $m_{u,v}$ has closed range.

Conversely if u^r is not bounded away from zero on $[Z(u)]^{'}$, then there exists a sequence $\{n_k\}$ of positive integers such that $|u(n_k)|^{\frac{1}{n_k}} = 0$. Define $h : N \to X$ as

$$h(m) = \begin{cases} 0, \text{ if } m \notin \{n_k : k \in N\}; \\ |u(n_k)|x, \text{if } m = n_k \text{ for some } k \in N. \end{cases}$$

Then $(p_u(h(m)))^{\frac{1}{m}} = (p_u(|u(n_k)|x))^{\frac{1}{n_k}} = |u(n_k)|^{\frac{1}{n_k}}(p_u(x))^{\frac{1}{n_k}} \to 0$ as $k \to \infty$ so that $h \in E(X)$. Let $g_m = \sum_{k=1}^{m} e(v(n_k))$. Then $P_u(m_{u,v}g_m - h) = \sup\{(p_u(u.g_m ov - h)(k))^{\frac{1}{k}}\} = 0$. Hence $h \in$ ran $m_{u,v}$ so that ran $m_{u,v}$ is closed. □

Theorem 2.4. *Let $m_{u,v} \in B(E(X))$. Then $m_{u,v}$ is compact if and only if the set $E_\epsilon = \{n \in N : |u(n)|^{\frac{1}{n}} \geq \epsilon\}$ is a finite set for each $\epsilon > 0$.*

Proof. Suppose that $m_{u,v}$ is a compact operator. We see that E_ϵ is a finite set for each $\epsilon > 0$. If E_ϵ is an infinite set, then we can choose a sequence $\{n_k\}$ of infinitely many elements such that $(u(n_k))^{\frac{1}{n_k}} \geq \epsilon$ and $v(n_k) \in E_\epsilon$. Now $\{e_{n_k}x\}$ is a bounded sequence in $E(X)$ and $P_u(m_{u,v}e_{n_i}x - m_{u,v}e_{n_j}x) \geq \epsilon(p_u(x))^{\frac{1}{n_k}}$ which does not go to zero. This contradicts the continuity of $m_{u,v}$. Hence the condition of the theorem must be true.

Conversely, assume that E_ϵ is a finite set for each $\epsilon > 0$ so that $0 < \epsilon < \sup|u(n)|^{\frac{1}{n}}$. Let $H = \{n : |u(n)|^{\frac{1}{n}} < \epsilon\}$ and $G = \{n : |u(n)|^{\frac{1}{n}} \geq \epsilon\}$. Then $v(G) = \{p_1, p_2, \cdots, p_n\}$. Let $\{f_n\}$ be a bounded sequence in $E(X)$. That is for each $u \in U(X)$, there exists $m_u > 0$ such that $p_u(f_n) \leq m_u$ for all $n = 1, 2, 3, \cdots$. We can choose a positive integer n_0 and a subsequence $\{g_n\}$ of $\{f_n\}$ such that $\sup(p_u(g_n(p_i) - g_m(p_i)))^{\frac{1}{k}} < \frac{\epsilon}{3||u||}$. Now if $k \in G$, then

$(p_u(u(k)g_n(v(k)) - u(k)g_m(v(k)))^{\frac{1}{k}} < \frac{\epsilon}{3}$. Further, if $k \in H$, then $P_u(m_{u,v}g_n - m_{u,v}g_m) = \sup\{(p_u(u(k)g_n(v(k)) - u(k)g_m(v(k))))^{\frac{1}{k}} : k \geq 1\} \leq \epsilon(p_u(g_n(v(k)) - g_m(v(k)))^{\frac{1}{k}} < \epsilon$. This shows that $m_{u,v}$ is a compact operator. □

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