International Journal of Applied Mathematics
Volume 25 No. 1 2012, 125-138

ASYMPTOTIC PROPERTIES OF SOLUTIONS OF HIGHER ORDER NONLINEAR GENERALIZED DIFFERENCE EQUATION

M. Maria Susai Manuel[1 §], G. Britto Antony Xavier[2],
D.S. Dilip[3], G. Dominic Babu[4]

[1]Department of Science and Humanities
R.M.D. Engineering College
Kavaraipettai, 601 206, Tamil Nadu, S. INDIA
e-mail: manuelmsm_03@yahoo.co.in

[2,3,4]Department of Mathematics
Sacred Heart College
Tirupattur, 635 601, Vellore District
Tamil Nadu, S. INDIA

Abstract: In this paper, we discuss the asymptotic behaviour of solutions of the generalized difference equation

$$\Delta_\ell^n u(k) + f(k, u(k), \Delta_\ell u(k), \ldots, \Delta_\ell^{n-1} u(k)) = 0, \quad k \in [0, \infty), \tag{1}$$

where the function f is defined on $[0,\infty) \times \mathbb{R}^n$ and ℓ is a positive real.

AMS Subject Classification: 39A12
Key Words: generalized difference equation, generalized difference operator

1. Introduction

The basic theory of difference equations is based on the operator Δ defined as $\Delta u(k) = u(k+1) - u(k),\ k \in \mathbb{N} = \{0, 1, 2, 3, \cdots\}$. Eventhough many authors (see [1], [12]-[16]) have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k),\ k \in \mathbb{R},\ \ell \in \mathbb{N}(1), \tag{2}$$

no significant progress has taken on this line. Recently, E. Thandapani, M.M.S.

Received: January 3, 2012

[§]Correspondence author

Manuel and G.B. Antony Xavier [7] considered the definition of Δ as given in (2) and developed the theory of difference equations in a different direction (see [7, 8]). For convenience, the operator Δ defined by (2) is labelled as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}, many interesting results and applications in number theory (see [7,10,11]) were obtained. By extending the study related to sequences of complex numbers and ℓ to be real, some new qualitative properties like rotatory, expanding, shrinking, spiral and weblike were analysed for the solutions of difference equations involving Δ_ℓ. The results obtained using Δ_ℓ can be found in (see [7-9]). The qualitative properties of lower order difference equations were studied by various authors. See for example [3,6,15]. Very few results on the study of the properties of solutions of higher order difference equations are available in the literature, for example [2, 14]. The above studies are based on the difference operator Δ. The results found in this paper generalize those of [2]. In this paper, we discuss various properties of the solutions for (1).

Throughout this paper we make use the following notations:

(i) $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, $\mathbb{N}(a) = \{a, a+1, a+2, \dots\}, \mathbb{N}_\ell(j) = \{j, j+\ell, j+2\ell, \dots\}$,

(ii) $[x]$ denotes the integer part of x and $\lceil x \rceil$ denotes upper integer part of x.

(iii) F_p denotes the class of all functions $u(k)$ defined on $[a, \infty)$ such that $|u(k)| = o\big((k)_\ell^{(p)}\big)$ as $k \to \infty$.

2. Preliminaries

In this section, we present some preliminary definitions and theorems which will be useful for further subsequent discussion.

Definition 2.1. (see [7]) Let $u(k)$, $k \in [0, \infty)$ be a real or complex valued function and $\ell \in (0, \infty)$. Then, the generalized difference operators Δ_ℓ and ∇_ℓ are defined as

$$\Delta_\ell u(k) = u(k+\ell) - u(k). \tag{3}$$

$$\nabla_\ell u(k) = u(k) - u(k-\ell). \tag{4}$$

Similarly, the generalized difference operators of the r^{th} kind are defined as

$$\Delta_\ell^r u(k) = \underbrace{\Delta_\ell(\Delta_\ell(\dots(\Delta_\ell u(k))))}_{r\ times}. \tag{5}$$

$$\nabla_\ell^r u(k) = \underbrace{\nabla_\ell(\nabla_\ell(\dots(\nabla_\ell u(k))))}_{r\ times}. \tag{6}$$

Definition 2.2. (see [7]) Let $u(k),\ k \in [0,\infty)$ be a real or complex valued function and $\ell \in (0,\infty)$. Then, the inverse of Δ_ℓ denoted by Δ_ℓ^{-1} is defined as follows.

$$\text{If } \Delta_\ell v(k) = u(k), \quad \text{then } v(k) = \Delta_\ell^{-1} u(k) + c_j, \tag{7}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j), j = k - \left[\frac{k}{\ell}\right]\ell$.

In general $\Delta_\ell^{-n} u(k) = \Delta_\ell^{-1}(\Delta_\ell^{-(n-1)} u(k))$ for $n \in \mathbb{N}(2)$.

Similarly one can define ∇_ℓ^{-n}.

The following is an extension of Lemma 4.2 in [7]

Lemma 2.3. (see [7]) *If the real valued function $u(k)$ is defined for all $k \in [a,\infty)$, then*

$$\Delta_\ell^{-1} u(k) = \sum_{r=1}^{\left[\frac{k-a}{\ell}\right]} u(k - r\ell) + c_j, \tag{8}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - a - \left[\frac{k-a}{\ell}\right]\ell$.

Theorem 2.4. *If $\Delta_\ell v(k) = u(k)$ for $k \in [k_2,\infty)$ and $j = k - k_2 - \left[\frac{k-k_2}{\ell}\right]\ell$, then*

$$v(k) - v(k_2 + j) = \sum_{r=0}^{\left[\frac{k-k_2-j-\ell}{\ell}\right]} u(k_2 + j + r\ell).$$

Proof. The proof follows from Definition 2.2, Lemma 2.3 and $c_j = v(k_2 + j)$. $\square$

Theorem 2.5. (see [8]) *Let $u(k)$ be defined on $[0,\infty)$ and $a \in [0,\infty)$. Then, for all $k \in [a,\infty)$, $j = k - a - \left[\frac{k-a}{\ell}\right]\ell$ and $0 \le m \le n-1$,*

$$\Delta_\ell^m u(k) = \sum_{i=m}^{n-1} \frac{(k-a)_\ell^{(i-m)}}{(i-m)!} \Delta_\ell^i u(a+j)$$

$$+ \sum_{r=0}^{\frac{k-a-j}{\ell} - n + m} \frac{(k - a - j - r\ell - \ell)_\ell^{(n-m-1)}}{(n-m-1)!} \Delta_\ell^n u(a + j + r\ell), \tag{9}$$

where $k_\ell^{(n)} = k(k-\ell)(k-2\ell)\dots(k-(n-1)\ell)$.

Lemma 2.6. (see [8]) *Let $u(k)$ and $v(k)$ be defined on $[a,\infty)$. Then, for all $k \in [a,\infty)$ and $j = k - a - \left[\frac{k-a}{\ell}\right]\ell$,*

$$\begin{aligned}&\Delta_\ell\{u(k)v(k)\}\\ &\quad = u(k+\ell)\Delta_\ell v(k) + v(k)\Delta_\ell u(k) = v(k+\ell)\Delta_\ell u(k) + u(k)\Delta_\ell v(k) \qquad (10)\end{aligned}$$

and

$$\begin{aligned}&\sum_{r=0}^{\frac{k-a-j-\ell}{\ell}} u(a+j+r\ell)\Delta_\ell v(a+j+r\ell)\\ &= u(r)v(r)\big|_{r=a}^{k} - \sum_{r=0}^{\frac{k-a-j-\ell}{\ell}} v(a+j+\ell+r\ell)\Delta_\ell u(a+j+r\ell). \qquad (11)\end{aligned}$$

Proof. By operating Δ_ℓ^{-1} on both sides of (10) and from Definition 2.2 and Theorem 2.4, (11) follows. □

Definition 2.7. (see [4]) Two difference systems are said to be asymptotically equivalent if, corresponding to each solution of one system, there exists a solution of the other system such that the difference between these two solutions converges to zero.

3. Main Results

Theorem 3.1. *Let for all $k \in [a,\infty), a \in [0,\infty)$ and $j = k - a - \left[\frac{k-a}{\ell}\right]\ell$ the following inequality be satisfied*

$$u(k) \le p(k) + q(k) \sum_{r=0}^{\frac{k-a-j-\ell}{\ell}} f(a+j+r\ell)u(a+j+r\ell). \qquad (12)$$

Then, for all $k \in [a,\infty)$ and $w(r,\tau) = 1 + q(r+\ell+j+\tau\ell)f(r+\ell+j+\tau\ell)$,

$$u(k) \le p(k) + q(k) \sum_{r=0}^{\frac{k-a-j-\ell}{\ell}} p(a+j+r\ell)f(a+j+r\ell) \prod_{\tau=0}^{\frac{k-r-j-2\ell}{\ell}} w(r,\tau). \qquad (13)$$

Proof. Define a function $v(k)$ on $[a,\infty)$ as follows

$$v(k)=\sum_{r=0}^{\frac{k-a-j-\ell}{\ell}} f(a+j+r\ell)u(a+j+r\ell).$$

For this function, we have

$$\Delta_\ell v(k)=f(k)u(k), v(a)=0. \tag{14}$$

Since $u(k)\le p(k)+q(k)v(k)$, and $f(k)\ge 0$, from (14) we obtain

$$v(k+\ell)-(1+q(k)f(k))v(k)\le p(k)f(k). \tag{15}$$

Let

$$e(q,f,r,a+j)=(1+q(a+j+r\ell)f(a+j+r\ell))^{-1}$$

and because

$$1+q(k)f(k)>0$$

for all $k\in[a,\infty)$, we can multiply (15) by $\prod_{r=0}^{\frac{k-a-j}{\ell}} e(q,f,r,a+j)$, to obtain

$$\Delta_\ell\left[\prod_{r=0}^{\frac{k-a-j-\ell}{\ell}} e(q,f,r,a+j)v(k)\right]\le p(k)f(k)\prod_{r=0}^{\frac{k-a-j}{\ell}} e(q,f,r,a+j).$$

From Lemmas 2.2 and 2.4 and using $v(a)=0$, we obtain

$$\prod_{r=0}^{\frac{k-a-j-\ell}{\ell}} e(q,f,r,a+j)v(k)\le \sum_{r=0}^{\frac{k-a-j-\ell}{\ell}} p(a+j+r\ell)f(a+j+r\ell)\prod_{\tau=0}^{r} e(q,f,\tau,a+j).$$

which is the same as

$$v(k)\le \sum_{r=0}^{\frac{k-a-j-\ell}{\ell}} p(a+j+r\ell)f(a+j+r\ell)\prod_{\tau=0}^{\frac{k-r-j-2\ell}{\ell}} w(r,\tau). \tag{16}$$

Now (13) follows from (16) and the inequality $u(k)\le p(k)+q(k)v(k)$. □

Corollary 3.2. *Let in Theorem 3.1, $p(k) = p$ and $q(k) = q$ for all $k \in [a, \infty)$. Then, for all $k \in [a, \infty)$ and $j = k - a - \left[\frac{k-a}{\ell}\right]\ell$,*

$$u(k) \leq p \prod_{r=0}^{\frac{k-a-j-\ell}{\ell}} (1 + qf(a + j + r\ell)).$$

Proof. We have from Lemma 2.4 and (16),

$$\Delta_\ell v(k) \leq pf(k) \prod_{r=0}^{\frac{k-a-j-\ell}{\ell}} (1 + qf(a + j + r\ell))$$

and the proof follows by (14). □

Lemma 3.3. *Let $1 \leq m \leq n-1$ and $u(k)$ be defined on $[a, \infty)$. Then,*

(i) $\lim_{k\to\infty} \inf \Delta_\ell^m u(k) > 0$ *implies* $\lim_{k\to\infty} \Delta_\ell^i u(k) = \infty, 0 \leq i \leq m-1$

(ii) $\lim_{k\to\infty} \sup \Delta_\ell^m u(k) < 0$ *implies* $\lim_{k\to\infty} \Delta_\ell^i u(k) = -\infty, 0 \leq i \leq m-1.$

Proof. $\lim_{k\to\infty} \inf \Delta_\ell^m u(k) > 0$ implies that there exists a large $k_1 \in [a, \infty)$ such that $\Delta_\ell^m u(k) \geq c > 0$ for all $k \in [k_1, \infty)$. Since

$$\Delta_\ell^{m-1} u(k) = \Delta_\ell^{m-1} u(k_1+j) + \sum_{r=0}^{\frac{k-k_1-j-\ell}{\ell}} \Delta_\ell^m u(k_1+j+r\ell) \text{ and } j = k-k_1-\left[\frac{k-k_1}{\ell}\right]\ell,$$

it follows that

$$\Delta_\ell^{m-1} u(k) \geq \Delta_\ell^{m-1} u(k_1 + j) + c(k - k_1 - j),$$

and hence

$$\lim_{k\to\infty} \Delta_\ell^{m-1} u(k) = \infty.$$

The rest of the proof is by induction. The case (ii) can be treated similarly. □

Theorem 3.4. *(Generalized Discrete l'Hospital's Rule).*
Let $u(k)$ and $v(k)$ be defined on $[a, \infty)$ and $v(k) > 0, \Delta_\ell v(k) < 0$ for all large k in $[a, \infty)$. If $\lim_{k\to\infty} u(k) = \lim_{k\to\infty} v(k) = 0$, then

$$\lim_{k\to\infty} \inf \frac{\Delta_\ell u(k)}{\Delta_\ell v(k)} \leq \lim_{k\to\infty} \inf \frac{u(k)}{v(k)} \leq \lim_{k\to\infty} \sup \frac{u(k)}{v(k)} \leq \lim_{k\to\infty} \sup \frac{\Delta_\ell u(k)}{\Delta_\ell v(k)}. \quad (17)$$

Proof. Let $k_1 \in [a, \infty)$ be sufficiently large so that for all $k \in [k_1, \infty), v(k) > 0$ and $\Delta_\ell v(k) < 0$. We assume that

$$\frac{\Delta_\ell u(k)}{\Delta_\ell v(k)} \geq c \text{ for all } k \in [k_1, \infty),$$

where $c \in [0, \infty)$. Then, $\Delta_\ell u(k) \leq c\Delta_\ell v(k)$ and by summation we obtain $u(k + p\ell) - u(k) \leq c(v(k + p\ell) - v(k))$ for all $k \in [k_1, \infty)$ and $0 < p \in [0, \infty)$. Allowing $p \to \infty$, we find $-u(k) \leq -cv(k)$, which is the same as $\frac{u(k)}{v(k)} \geq c$ for all $k \in [k_1, \infty)$. Since the same holds with the inequalities reversed, (17) holds. □

Corollary 3.5. *Let $u(k)$ and $v(k)$ be as in Theorem 3.4. Then,*

$$\lim_{k\to\infty} \frac{u(k)}{v(k)} = c \text{ provided } \lim_{k\to\infty} \frac{\Delta_\ell u(k)}{\Delta_\ell v(k)} = c \text{ exists.}$$

Example 3.6. $u(k) = \frac{1}{k_\ell^{(3)}}, v(k) = \frac{1}{k_\ell^{(4)}}$ satisfies the conditions of Corollary 3.5.

Theorem 3.7. *Assume that the function $f(k, u_o, \ldots, u_{n-1})$ satisfies*

$$|f(k, u_o, \ldots, u_{n-1})| \leq \sum_{i=0}^{n-1} p_i(k)|u_i|, \text{ for all } f(k, u_o, \ldots, u_{n-1}) \in [0, \infty) \times \mathbb{R}^n, \tag{18}$$

where $p_i(k), 0 \leq i \leq n-1$ are nonnegative functions, defined on $[0, \infty)$ and

$$\prod_{r=0}^{\infty} \left[1 + \sum_{i=0}^{n-1} (a + j + r\ell)_\ell^{(n-i-1)} p_i(a + j + r\ell)\right] < \infty \text{ where, } j = k - a - \left[\frac{k-a}{\ell}\right]\ell. \tag{19}$$

Then, the difference equation (1) has solutions which are asymptotic to $\sum_{i=0}^{n-1} a_i(k)_\ell^{(i)}$ as $k \to \infty$, where $a_i, 0 \leq i \leq n-1$ are constants such that $a_{n-1} \neq 0$.

Proof. Let $u(k)$ be a solution of (1). For any $a \in [\ell, \infty)$, Theorem 2.5 yields

$$\Delta_\ell^m u(k) = \sum_{i=m}^{n-1} \frac{(k-a)_\ell^{(i-m)}}{(i-m)!} \Delta_\ell^i u(a+j)$$

$$-\frac{1}{(n-m-1)!} \sum_{r=0}^{\frac{k-a-j}{\ell}-n+m} (k - a - j - r\ell - \ell)_\ell^{(n-m-1)}$$

$$\times f(a+j+r\ell, u(a+j+r\ell), \Delta_\ell u(a+j+r\ell), \dots, \Delta_\ell^{n-1} u(a+j+r\ell)), 0 \le m \le n-1. \tag{20}$$

Thus, from (18) we find $|\Delta_\ell^m u(k)| \le A_m (k)_\ell^{(n-m-1)}$

$$+B_m (k)_\ell^{(n-m-1)} \sum_{r=0}^{\frac{k-a-j}{\ell}-n+m} \sum_{i=0}^{n-1} p_i(a+j+r\ell)|\Delta_\ell^i u(a+j+r\ell)|, \tag{21}$$

where $A_m = \left[(k)_\ell^{(n-m-1)}\right]^{-1} \sum_{i=m}^{n-1} \frac{(k)_\ell^{(i-m)}}{(i-m)!} |\Delta_\ell^i u(a+j)|$, and $B_m = 1/(n-m-1)!$. Define $A = \max_{0 \le m \le n-1, k \in [a,\infty)} A_m$, then, since $B_m \le 1, 0 \le m \le n-1$ from (21), we obtain

$$|\Delta_\ell^m u(k)| \le (k)_\ell^{(n-m-1)} F(k), 0 \le m \le n-1, \tag{22}$$

where

$$F(k) = A + \sum_{r=0}^{\frac{k-a-j}{\ell}-n+m} \sum_{i=0}^{n-1} p_i(a+j+r\ell)|\Delta_\ell^i u(a+j+r\ell)|.$$

Using (22) in the above equality, we obtain

$$F(k) \le A + \sum_{r=0}^{\frac{k-a-j-\ell}{\ell}} \sum_{i=0}^{n-1} (a+j+r\ell)_\ell^{(n-i-1)} p_i(a+j+r\ell) F(a+j+r\ell).$$

Application of Corollary 3.2, yields

$$F(k) \le A \prod_{r=0}^{\frac{k-a-j-\ell}{\ell}} \left[1 + \sum_{i=0}^{n-1} (a+j+r\ell)_\ell^{(n-i-1)} p_i(a+j+r\ell)\right]$$

and hence from (19) there exists a finite constant $c > 0$ such that $F(k) \le c$. Thus, inequality (22) implies that

$$|\Delta_\ell^m u(k)| \le c(k)_\ell^{(n-m-1)}, 0 \le m \le n-1. \tag{23}$$

Also, from (20) we have $\Delta_\ell^{n-1} u(k) = \Delta_\ell^{n-1} u(a+j)$

$$- \sum_{r=0}^{\frac{k-a-j-\ell}{\ell}} f(a+j+r\ell, u(a+j+r\ell), \Delta_\ell u(a+j+r\ell), \dots, \Delta_\ell^{n-1} u(a+j+r\ell)). \tag{24}$$

Since condition (19) implies that $\sum_{r=0}^{\infty}\sum_{i=0}^{n-1}(a+j+r\ell)_\ell^{(n-i-1)}p_i(a+j+r\ell) < \infty$, we find from (18) and (23) that the sum in (24) converges as $k \to \infty$ and therefore $\lim\limits_{k\to\infty} \Delta_\ell^{n-1}u(k)$ exists and is a finite number. To ensure that this limit is not zero, we choose a so large that $1 - c\sum_{r=0}^{\infty}\sum_{i=0}^{n-1}(a+j+r\ell)_\ell^{(n-i-1)}p_i(a+j+r\ell) > 0$ and impose the condition $\Delta_\ell^{n-1}u(a) = 1$ on the solution of (1). This solution has the desired asymptotic property. □

Example 3.8. For the generalized difference equation $\Delta_\ell^3 u(k) = 6\ell^3 - \frac{6\ell^3}{(k+\ell)_\ell^{(6)}}$ and for $p_i(k) = \dfrac{(3-i)!\ell^{3-i}}{k_\ell^{(3-i)}}$, the solution is asymptotic to $\sum\limits_{i=0}^{2} a_i(k)_\ell^{(i)}$ as $k \to \infty$, where $a_i, 0 \le i \le 2$ are constants. Infact $u(k) = k_\ell^{(3)} + \frac{1}{k_\ell^{(3)}}$, is a solution of the difference equation.

Corollary 3.9. *Under the hypotheses of Theorem 3.7, equation (1) has all solutions nonoscillatory.*

Theorem 3.10. *If there exists a constant $c > 0$ such that, for any function $u(k)$ defined on $[0,\infty)$, $\liminf\limits_{k\to\infty} u(k) > c$ ($\limsup\limits_{k\to\infty} u(k) < -c$) and $j = k - a - \left[\frac{k-a}{\ell}\right]\ell$ such that*

$$\sum_{r=0}^{\infty} f(a+j+r\ell, u(a+j+r\ell), \Delta_\ell u(a+j+r\ell), \ldots, \Delta_\ell^{n-1}u(a+j+r\ell)) = \pm\infty, \quad (25)$$

then every nonoscillatory solution $u(k)$ of (1) satisfies $\liminf\limits_{k\to\infty} |u(k)| \le c$.

Proof. Let $u(k)$ be a nonoscillatory solution of (1), say $u(k) > 0$ for all $k \ge a$, and assume that $\liminf\limits_{k\to\infty} u(k) > c$. The case $u(k) < 0$ for all $k \ge a$ can be treated similarly. From (24) and (25) it is clear that $\lim\limits_{k\to\infty} \Delta_\ell^{n-1}u(k) = -\infty$, and therefore $\limsup\limits_{k\to\infty} \Delta_\ell^{n-1}u(k) < 0$. But, then, Lemma 3.3 implies that $\lim\limits_{k\to\infty} u(k) = -\infty$, which is a contradiction to our assumption that $u(k) > 0$. □

Example 3.11. For the generalized difference equation $\Delta_\ell^4 u(k) = 5!\ell^4(3k - 42k_\ell^{(3)})$ every nonoscillatory solution $u(k)$ of (1) satisfies the condition of Theorem 3.10.

Lemma 3.12. *Consider the difference equation*

$$\nabla_\ell u(k) - \frac{m}{k}u(k) + \frac{f(k)}{k} = 0, k \in [m, \infty), m \in \mathbb{N}(1), \tag{26}$$

where the function f is defined on $[m,\infty)$ and nonoscillatory. If $\lim_{k\to\infty} |f(k)| = \infty$ and $u(k)$ is the solution of (26) with $u(a) = 0$, where $m < a \in [m,\infty)$, then $\lim_{k\to\infty} u(k) = \pm\infty$.

Proof. By direct substitution, it is easy to verify that, for $j = k-a-\left[\frac{k-a}{\ell}\right]\ell$,

$$u(k) = -(k)_\ell^{(m)} \sum_{r=0}^{\frac{k-a-j-\ell}{\ell}} f(a+j+\ell+r\ell)/(a+j+\ell+r\ell)_\ell^{(m+1)} \tag{27}$$

is a solution of (26) satisfying $u(a) = 0$. Since f is of constant sign for all large k, the summation

$$\sum_{r=0}^{\infty} f(a+j+r\ell)/(a+j+r\ell)_\ell^{(m+1)}$$

exists on the extended real line. If the value of this summation is different from zero, then the result is trivial. If it is zero, then, let

$$p(k) = \sum_{r=0}^{\frac{k-a-j-\ell}{\ell}} f(a+j+\ell+r\ell)/(a+j+\ell+r\ell)_\ell^{(m+1)}$$

and

$$q(k) = 1/(k)_\ell^{(m)}$$

so that

$$\Delta_\ell p(k) = \frac{f(k+\ell)}{(k+\ell)_\ell^{(m+1)}}$$

and

$$\Delta_\ell q(k) = \frac{-m\ell}{(k+\ell)_\ell^{(m+1)}}.$$

By Corollary 3.5 we find that

$$\lim_{k\to\infty} \frac{p(k)}{q(k)} = \lim_{k\to\infty} \frac{\Delta_\ell p(k)}{\Delta_\ell q(k)} = \lim_{k\to\infty} \frac{f(k+\ell)}{-m\ell} = \pm\infty.$$

Therefore, $\lim_{k\to\infty} u(k) = \pm\infty$. □

Theorem 3.13. *Assume that there exist integers p, q, τ such that $0 \le \tau \le n-1$, $0 \le q \le p \le n-\tau-1$, and for every nonoscillatory $u(k) \in F_p$ with*

$$\lim_{k\to\infty} \inf |u(k)|/(k)_\ell^{(q)} \neq 0$$

and $j = k - a - \left[\frac{k-a}{\ell}\right]\ell$,

$$\sum_{r=0}^{\infty} (a+j+r\ell)_\ell^{(\tau)}$$

$$\times f(a+j+r\ell, u(a+j+r\ell), \Delta_\ell u(a+j+r\ell), \ldots, \Delta_\ell^{n-1} u(a+j+r\ell)) = \pm\infty. \quad (28)$$

Then, for all nonoscillatory F_p solutions $u(k)$ of (1), $\lim_{k\to\infty} \inf |u(k)|/(k)_\ell^{(q)} = 0$.

Proof. Let $u(k)$ be a nonoscillatory F_p- solution with $\lim_{k\to\infty} \inf |u(k)|/(k)_\ell^{(q)} \neq 0$. Without loss of generality, we assume that $u(k) > 0$ on $[k_1, \infty)$, where $k_1 > \max\{1, \tau\}$. The case $u(k) < 0$ can be treated similarly. We define

$$R_{ih}^s(k) = \sum_{r=0}^{\frac{k-k_1-j-\ell}{\ell}} (k_1+j+r\ell)_\ell^{(i)} \Delta_\ell^h u(k_1+j+r\ell+s),$$

where $j = k-k_1-\left[\frac{k-k_1}{\ell}\right]\ell$ and from (11) find $R_{ih}^s(k) = (r)_\ell^{(i)} \Delta_\ell^{h-1} u(r+s)\Big|_{r=k_1+j}^{k}$

$$-i\ell \sum_{r=0}^{\frac{k-k_1-j-\ell}{\ell}} (k_1+j+r\ell)_\ell^{(i-1)} \Delta_\ell^{h-1} u(k_1+j+r\ell+s+\ell).$$

Since

$$\nabla_\ell R_{i-1,h-1}^{s+\ell}(k) = (k-\ell)_\ell^{(i-1)} \Delta_\ell^{h-1} u(k+s),$$

the above equation takes the form

$$R_{ih}^s(k) = k\nabla_\ell v(k) - (k_1+j)_\ell^{(i)} \Delta_\ell^{h-1} u(k_1+j+s) - i\ell v(k),$$

where $v(k) = R_{i-1,h-1}^{s+\ell}(k)$. Thus, we find that

$$\nabla_\ell v(k) - \frac{i\ell}{k} v(k) + \frac{f_{ih}^s(k)}{k} = 0, v(k_1) = 0, k_1 > \tau, \quad (29)$$

where $f^s_{ih}(k) = -(k_1+j)^{(i)}_\ell \Delta^{h-1}_\ell u(k_1+j+s) - R^s_{ih}(k)$. Let $i=\tau, h=n$ and $s=0$, then from (1) we have

$$f^0_{\tau n}(k) = -(k_1+j)^{(\tau)}_\ell \Delta^{n-1}_\ell u(k_1+j) + \sum_{r=0}^{\frac{k-k_1-j-\ell}{\ell}} (k_1+j+r\ell)^{(\tau)}_\ell$$

$$\times f(k_1+j+r\ell, u(k_1+j+r\ell), \Delta_\ell u(k_1+j+r\ell), \ldots, \Delta^{n-1}_\ell u(k_1+j+r\ell))$$

and from (28), $f^0_{\tau n}(k)$ is nonoscillatory and $\lim_{k\to\infty} |f^0_{\tau n}(k)| = \infty$. Thus, from Lemma 3.12, we obtain

$$\lim_{k\to\infty} v(k) = \lim_{k\to\infty} R^\ell_{\tau-1,n-1}(k) = \pm\infty. \tag{30}$$

Also, since

$$f^\ell_{\tau-1,n-1}(k) = -(k_1+j)^{(\tau-1)}_\ell \Delta^{n-2}_\ell u(k_1+j+\ell) - R^\ell_{\tau-1,n-1}(k),$$

from (30), we find $f^\ell_{\tau-1,n-1}(k)$ as nonoscillatory and $\lim_{k\to\infty} |f^\ell_{\tau-1,n-1}(k)| = \infty$. Thus, by Lemma 3.12 we obtain

$$\lim_{k\to\infty} R^{2\ell}_{\tau-2,n-2}(k) = \pm\infty.$$

Continuing this way, we find that

$$\lim_{k\to\infty} R^{\tau\ell}_{0,n-\tau}(k) = \pm\infty.$$

However, from the definition

$$R^{\tau\ell}_{0,n-\tau}(k) = \Delta^{n-\tau-1}_\ell u(k+\tau) - \Delta^{n-\tau-1}_\ell u(k_1+j+\tau),$$

we obtain

$$\lim_{k\to\infty} \Delta^{n-\tau-1}_\ell u(k) = \pm\infty.$$

The case

$$\lim_{k\to\infty} \Delta^{n-\tau-1}_\ell u(k) = -\infty$$

is impossible by Lemma 3.3, since it is a contradiction to the fact that $u(k)$ is positive, we obtain

$$\lim_{k\to\infty} \Delta^{n-\tau-1}_\ell u(k) = \infty.$$

Since $u(k) > 0$ which belongs to F_p, there exists a constant $c > 0$ such that $u(k) < c(k)_\ell^{(p)}$ for large $k \in [0, \infty)$. Thus, the function $w(k) = u(k) - c(k)_\ell^{(p)}$ is negative for large $k \in [0, \infty)$. But, since $p \le n - \tau - 1$, we find

$$\lim_{k\to\infty} \Delta_\ell^{n-\tau-1} w(k) = \lim_{k\to\infty} \Delta_\ell^{n-\tau-1}\big(u(k) - c(k)_\ell^{(p)}\big) = \infty,$$

which from Lemma 3.3 leads to a contradiction that $w(k)$ is negative. This completes the proof. □

Example 3.14. For the difference equation $\Delta_\ell^5 u(k) = -\frac{8!\ell^5}{6(k+5\ell)_\ell^{(9)}} + \frac{7!\ell^5}{2} k_\ell^{(2)}$, (28) holds and $u(k)$ is F_7 solution then $\lim_{k\to\infty} \inf |u(k)|/(k)_\ell^{(q)} = 0$.

Remark 3.15. If $p = 0$ in Theorem 3.13, we have that, for all bounded nonoscillatory solutions of (1), $\lim_{k\to\infty} \inf |u(k)| = 0$.

Acknowledgments

Research Supported by National Board for Higher Mathematics, Department of Atomic Energy, Government of India, Mumbai.

References

[1] R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York (2000).

[2] R.P. Agarwal, Properties of solutions of higher order nonlinear difference equations, *An. St. Univ. Iasi*, **29** (1983), 85-96.

[3] R.P. Agarwal, M.M.S. Manuel, E. Thandapani, Oscillatory and nonoscillatory behaviour of second order neutral delay difference equations, *Math. Comput. Modelling*, **24**, No. 1 (1996), 5-11.

[4] L.G. Lukianov, On the plane form and symmetry of asymptotic solutions in the restricted three-body problem, *Soviet Astronomy*, **22** (1978), 88-91.

[5] Blazej Szmanda, Nonoscillation, oscillation and growth of solutions of nonlinear difference equations of second order, *Journal of Mathematical Analysis and Applications*, **109**, No. 1 (1985), 22-30.

[6] B.S. Lalli, B.G. Zhang, J.Z. Li, On the oscillation of solutions and existence of positive solutions of neural difference equations, *J. Math. Anal. Appl.*, **158**, No. 1 (1991), 213-233.

[7] M. Maria Susai Manuel, G. Britto Antony Xavier, E. Thandapani, Theory of generalized difference operator and its applications, *Far East Journal of Mathematical Sciences*, **20**, No. 2 (2006), 163-171.

[8] M. Maria Susai Manuel, G. Britto Antony Xavier, Generalized difference calculus of sequences of real and complex numbers, *International Journal of Computational and Numerical Analysis and Applications*, **6**, No. 4 (2004), 401-415.

[9] M. Maria Susai Manuel, G. Britto Antony Xavier, E. Thandapani, Qualitative properties of solutions of certain class of difference equations, *Far East Journal of Mathematical Sciences*, **23**, No. 3 (2006), 295-304.

[10] R. Pugalarasu, M. Maria Susai Manuel, V. Chandrasekar, G. Britto Antony Xavier, Theory of generalized difference operator of n-th kind and its applications in number theory (Part I), *International Journal of Pure and Applied Mathematics*, **64**, No. 1 (2010), 103-120.

[11] R. Pugalarasu, M. Maria Susai Manuel, V. Chandrasekar, G. Britto Antony Xavier,Theory of generalized difference operator of n-th king and its applications in number theory (Part II), *International Journal of Pure and Applied Mathematics*, **64**, No. 1 (2010), 121-132.

[12] Ronald E. Mickens, *Difference Equations*, Van Nostrand Reinhold Company, New York (1990).

[13] Saber N. Elaydi, *An Introduction to Difference Equations*, Second Edition, Springer (1999).

[14] E. Thandapani, M. Maria Susai Manuel, John R. Graef, Paul W. Spikes, Oscillation of a higher order neutral difference equation with a forcing term, *International Journal of Math. and Math. Sci.*, **22**, No. 1 (1999), 147-154.

[15] E. Thandapani, Oscillation theorems for perturbed nonlinear second order difference equations, *Comput. Math. Appl.*, **28**, No-s: 1-3 (1994), 309-316.

[16] Walter G. Kelley, Allan C. Peterson, *Difference Equations. An Introduction with Applications*, Academic Press (1991).