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SOME ESTIMATES FOR THE JACOBI TRANSFORM IN THE SPACE $\mathrm{L}^2(\mathbb{R}^+, \Delta_{(\alpha,\beta)}(t)dt)$

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Abstract: For the Jacobi transform in the space $\mathrm{L}^2(\mathbb{R}^+, \Delta_{(\alpha,\beta)}(t)dt)$, two useful estimates are proved in certain class of functions characterized by a generalized continuity modulus, using a generalized translation operator.

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1. Introduction

The Jacobi analysis can be developed as a generalized of the Fourier-cosine transform and has been studied by many authors (see [7]), the main interest being the interplay between the analytic and geometric properties of the Jacobi operator. Indeed, in certain cases, the Jacobi operator is the radial part of the Laplace-Beltrami operator on Damek-Ricci spaces [3], therefore the Jacobi analysis includes radial analysis on symmetric spaces of real rank one as a special.

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In this paper, we prove two useful estimates in certain classes of functions characterized by a generalized continuity modulus and connected with the Jacobi transform in the space $L^2(\mathbb{R}^+, \Delta_{(\alpha,\beta)}(t)dt)$, analogous of the statements proved in [2]. For this purpose, we use a generalized translation operator which was defined by Flensted-Jensen and Koornwinder [6].

In Section 2, we give some definitions and preliminaries concerning the Jacobi transform. The estimates are proved in Section 3.

2. The Jacobi Transform and its Basic Properties

Now, we collect some basic facts on the Jacobi transform, and more details about the Jacobi transform can be found in [3] and [7].

The Jacobi function $\phi_\lambda^{(\alpha,\beta)}(t)$ of order (α,β) $(\alpha \neq -1, -2, ...)$ is the unique C^∞ function on $\mathbb{R}$ which equals 1 at 0 and satisfies the differential equation

$$(\mathrm{D}_{\alpha,\beta} + \lambda^2 + \rho^2)\phi_\lambda^{(\alpha,\beta)}(t) = 0,$$

where $\lambda \in \mathbb{C}$, $\rho = \alpha + \beta + 1$ and

$$\mathrm{D}_{\alpha,\beta} = \frac{d^2}{dt^2} + ((2\alpha+1)coth t + (2\beta+1)tanh t\frac{d}{dt}.$$

Throughout this paper, we assume that $\alpha \geq \beta \geq \frac{-1}{2}$, and $\alpha > \frac{-1}{2}$.

Lemma 1. *Let $\alpha > \frac{-1}{2}$, $\alpha \geq \beta \geq \frac{-1}{2}$, and let $t_0 > 0$. Then for $|\eta| \leq \rho$, there exists a positive constant $C_1 = C_1(t_0, \alpha, \beta)$ such that*

$$|1 - \phi_{\mu+i\eta}^{(\alpha,\beta)}(t)| \geq C_1|1 - j_\alpha(\mu t)|, \tag{1}$$

where $j_\alpha(t)$ is a normalized Bessel function of the first kind.

Proof. See [4, Lemma 9]. □

Lemma 2. *The following inequalities are valid for the Jacobi function $\phi_\lambda^{(\alpha,\beta)}(t)$ $(\lambda, t \in \mathbb{R}^+)$:*

1. $|\phi_\lambda^{(\alpha,\beta)}(t)| \leq 1,$
2. $|1 - \phi_\lambda^{(\alpha,\beta)}(t)| \leq t^2(\lambda^2 + \rho^2).$

Proof. See [8, Lemmas 3.1-3.2]. $\square$

Consider the Hilbert space $\mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+) = \mathrm{L}^2(\mathbb{R}^+, \Delta_{(\alpha,\beta)}(t)dt)$ with the norm

$$\|f\| = \|f\|_{2,(\alpha,\beta)} = \left(\int_0^\infty |f(x)|^2 \Delta_{(\alpha,\beta)}(x)dx\right)^{1/2},$$

where $\Delta_{(\alpha,\beta)}(t) = (2sinht)^{2\alpha+1}(2cosht)^{2\beta+1}$.

The Jacobi transform of a function $f \in \mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$ is defined by

$$g(\lambda) = \int_0^\infty f(t)\phi_\lambda^{(\alpha,\beta)}(t)\Delta_{(\alpha,\beta)}(t)dt.$$

The inversion formula (cf. [7]) is

$$f(t) = \frac{1}{2\pi}\int_0^\infty g(\lambda)\phi_\lambda^{(\alpha,\beta)}(t)d\mu(\lambda),$$

where $d\mu(\lambda) = |C(\lambda)|^{-2}d\lambda$ and the C-function $C(\lambda)$ is defined by

$$C(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+\alpha+\beta+1))\Gamma(\frac{1}{2}(i\lambda+\alpha-\beta+1))}.$$

We have the following estimate for $C(\lambda)$ (cf. Corollary 9 in [5])

$$|C(\lambda)|^{-2} \sim (1+|\lambda|)^{2\alpha+1}, \quad \text{as} \quad \lambda \longrightarrow \infty.$$

The Plancherel formula for the Jacobi transform is written as

$$\|f\| = \|f\|_{\mathrm{L}^2(\mathbb{R}^+,\Delta_{(\alpha,\beta)}(t)dt)} = \|g\|_{\mathrm{L}^2(\mathbb{R}^+,\frac{1}{2\pi}d\mu(\lambda))}.$$

The Jacobi function $\phi_\lambda^{(\alpha,\beta)}(t)$ can be expressed by using the Gauss hypergeomtric function as

$$\phi_\lambda^{(\alpha,\beta)}(t) = \mathrm{F}(\frac{1}{2}(\rho-i\lambda), \frac{1}{2}(\rho+i\lambda), \alpha+1, -sinh^2t).$$

In this paper, we estimate the integral

$$\int_{\lambda\geq N} |g(\lambda)|^2 d\mu(\lambda)$$

in certain classes of functions in $\mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$.

Recall from [6, Formula (5.1)] the generalized translation T_h of a suitable function f on $\mathbb{R}^+$, defined by

$$\mathrm{T}_h f(x) = \int_0^\infty f(z)K(x,h,z)\Delta_{(\alpha,\beta)}(z)dz,$$

where K is an explicity known kernel function such that

$$\phi_\lambda^{(\alpha,\beta)}(x)\phi_\lambda^{(\alpha,\beta)}(y) = \int_0^\infty \phi_\lambda^{(\alpha,\beta)}(z)K(x,h,z)\Delta_{(\alpha,\beta)}(z)dz,$$

where

$$\begin{aligned} K(x,y,z) &= \frac{2^{-2\rho}\Gamma(\alpha+1)(coshx\ coshy\ coshz)^{-\alpha-\beta-1}}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})(sinhx\ sinhy\ sinhz)^{2\alpha}}(1-B^2)^{\alpha-\frac{1}{2}} \\ &\times\ F((\alpha+\beta,\alpha-\beta,\alpha+\frac{1}{2},\frac{1}{2}(1-B)) \end{aligned}$$

for $|x-y| < z < x+y$ and $K(x,y,z) = 0$ elsewhere, and

$$B = \frac{cosh^2x + cosh^2y + cosh^2z - 1}{2coshx\ coshy\ coshz}.$$

The finite differences of the first and higher orders are defined as follows:

$$\Delta_h f(x) = \mathrm{T}_h f(x) - f(x) = (\mathrm{T}_h - \mathrm{E})f(x),$$

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1}f(x)) = (\mathrm{T}_h - \mathrm{E})^k f(x) = \sum_{i=0}^{k}(-1)^{k-1}\binom{k}{i}\mathrm{T}_h^i f(x),$$

where $\mathrm{T}_h^0 f(x) = f(x)$, $\mathrm{T}_h^i f(x) = \mathrm{T}_h(\mathrm{T}_h^{i-1}f(x))$ $(i = 1,2,...,k$ and $k = 1,2,...)$, E is a unit operator in $\mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$, and

$$\Omega_k(f,\delta) = \sup_{0<h\le\delta}\|\Delta_h^k f(x)\|$$

is the k-th order generalized continuity modulus $f \in \mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$.

Denote by $\mathrm{W}^{r,k}_{2,\psi}(\mathrm{D}_{\alpha,\beta})$ the class of functions $f \in \mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$ such that

$$\Omega_k(\mathrm{D}^r_{\alpha,\beta}f,\delta) = O(\psi(\delta^k))$$

and $\psi(t)$ is an arbitrary function defined on $[0,\infty)$.

Since in [4]

$$\mathrm{T}_h f(x) = \frac{1}{2\pi}\int_0^\infty \phi_\lambda^{(\alpha,\beta)}(h)g(\lambda)\phi_\lambda^{(\alpha,\beta)}(x)d\mu(\lambda)$$

and

$$f(x) = \frac{1}{2\pi}\int_0^\infty g(\lambda)\phi_\lambda^{(\alpha,\beta)}(x)d\mu(\lambda),$$

it follows

$$\mathrm{T}_h f(x) - f(x) = \frac{1}{2\pi}\int_0^\infty (1-\phi_\lambda^{(\alpha,\beta)}(h))g(\lambda)\phi_\lambda^{(\alpha,\beta)}(x)d\mu(\lambda). \tag{2}$$

The Plancherel equality and formula (2) give

$$\|\mathrm{T}_h f(x) - f(x)\|^2 = \int_0^\infty |1-\phi_\lambda^{(\alpha,\beta)}(h)|^2 |g(\lambda)|^2 d\mu(\lambda).$$

Hence, for $f \in \mathrm{W}^{r,k}_{2,\psi}(\mathrm{D}_{\alpha,\beta})$, we have

$$\|\Delta_h^k \mathrm{D}^r_{\alpha,\beta} f(x)\|^2 = \int_0^\infty (\lambda^2+\rho^2)^{2r}|1-\phi_\lambda^{(\alpha,\beta)}(h)|^{2k}|g(\lambda)|^2 d\mu(\lambda). \tag{3}$$

3. Estimates for the Jacobi Transform

Taking into account the preliminaries in Section 2, for some classes of functions characterized by the generalized modulus of continuity, we can prove two estimates for the integral

$$\int_{\lambda\geq N} |g(\lambda)|^2 d\mu(\lambda).$$

Theorem 1. *For functions $f(x) \in \mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$ in the class $\mathrm{W}^{r,k}_{2,\psi}(\mathrm{D}_{\alpha,\beta})$,*

$$\sup_{\mathrm{W}^{r,k}_{2,\psi}(\mathrm{D}_{\alpha,\beta})} \sqrt{\int_{\lambda\geq N} |g(\lambda)|^2 d\mu(\lambda)} = O(N^{-2r}\psi((\frac{c}{N})^k),$$

where $r = 0,1,...$, $k = 1,2,...$, $c > 0$ is a fixed constant, and $\psi(t)$ is any function defined on the interval $[0,\infty)$

Proof. In the terms of $j_p(x)$, the normalized Bessel function of the first kind, we have (see [1])

$$1 - j_p(x) = O(1),\ x \geq 1, \tag{4}$$

$$1 - j_p(x) = O(x^2),\ 0 \leq x \leq 1, \tag{5}$$

$$\sqrt{hx}J_p(hx) = O(1),\ hx \geq 0, \tag{6}$$

where $J_p(x)$ is Bessel function of the first kind, and

$$j_p(x) = \frac{2^p\Gamma(p+1)}{x^p}J_p(x). \tag{7}$$

Let $f \in \mathrm{W}_{2,\psi}^{r,k}(\mathrm{D}_{\alpha,\beta})$. Taking into account the Hölder inequality yields

$$\int_{\lambda\geq N}|g(\lambda)|^2d\mu(\lambda) - \int_{\lambda\geq N}|g(\lambda)|^2 j_\alpha(\lambda h)d\mu(\lambda) = \int_{\lambda\geq N}(1-j_\alpha(\lambda h)|g(\lambda)|^2d\mu(\lambda)$$

$$= \int_{\lambda\geq N}(1-j_\alpha(\lambda h))(|g(\lambda)|\frac{1}{|C(\lambda)|})^{2-\frac{1}{k}}(|g(\lambda)|\frac{1}{|C(\lambda)|})^{\frac{1}{k}}d\lambda$$

$$\leq \left(\int_{\lambda\geq N}|g(\lambda)|^2d\mu(\lambda)\right)^{\frac{2k-1}{2k}}\left(\int_{\lambda\geq N}|1-j_\alpha(\lambda h)|^{2k}|g(\lambda)|^2d\mu(\lambda)\right)^{\frac{1}{2k}}$$

$$\leq \frac{1}{C_1}\left(\int_{\lambda\geq N}|g(\lambda)|^2d\mu(\lambda)\right)^{\frac{2k-1}{2k}}\left(\int_{\lambda\geq N}|1-\phi_\lambda^{(\alpha,\beta)}(h)|^{2k}|g(\lambda)|^2d\mu(\lambda)\right)^{\frac{1}{2k}}$$

$$= \frac{1}{C_1}\left(\int_{\lambda\geq N}|g(\lambda)|^2d\mu(\lambda)\right)^{\frac{2k-1}{2k}}$$

$$\times\left(\int_{\lambda\geq N}(\lambda^2+\rho^2)^{-2r}|1-\phi_\lambda^{(\alpha,\beta)}(h)|^{2k}|g(\lambda)|^2(\lambda^2+\rho^2)^{2r}d\mu(\lambda)\right)^{\frac{1}{2k}}$$

$$\leq \frac{1}{C_1}(N^2+\rho^2)^{\frac{-r}{k}}$$

$$\times\left(\int_{\lambda\geq N}|g(\lambda)|^2d\mu(\lambda)\right)^{\frac{2k-1}{2k}}\left(\int_{\lambda\geq N}(\lambda^2+\rho^2)^{2r}|1-\phi_\lambda^{(\alpha,\beta)}(h)|^{2k}|g(\lambda)|^2d\mu(\lambda)\right)^{\frac{1}{2k}}.$$

From (3), we have

$$\int_{\lambda\geq N}(\lambda^2+\rho^2)^{2r}|1-\phi_\lambda^{(\alpha,\beta)}(h)|^{2k}|g(\lambda)|^2d\mu(\lambda) \leq \|\Delta_h^k\mathrm{D}_{\alpha,\beta}^r f(x)\|^2.$$

Therefore

$$\int_{\lambda\geq N}|g(\lambda)|^2 d\mu(\lambda)\leq\int_{\lambda\geq N}|g(\lambda)|^2 j_\alpha(\lambda h)d\mu(\lambda)$$
$$+\frac{1}{C_1}(N^2+\rho^2)^{\frac{-r}{k}}\left(\int_{\lambda\geq N}|g(\lambda)|^2 d\mu(\lambda)\right)^{\frac{2k-1}{2k}}\|\Delta_h^k \mathrm{D}_{\alpha,\beta}^r f(x)\|^{1/k}.$$

Combining this with formulas (6) and (7), we have

$$\int_{\lambda\geq N}|g(\lambda)|^2 d\mu(\lambda)=O\left(\int_{\lambda\geq N}(\lambda h)^{-\alpha-\frac{1}{2}}|g(\lambda)|^2 d\mu(\lambda)\right)$$
$$+N^{\frac{-2r}{k}}\left(\int_{\lambda\geq N}|g(\lambda)|^2 d\mu(\lambda)\right)^{\frac{2k-1}{2k}}\|\Delta_h^k \mathrm{D}_{\alpha,\beta}^r f(x)\|^{1/k}$$
$$=O(Nh)^{-\alpha-\frac{1}{2}}\int_{\lambda\geq N}|g(\lambda)|^2 d\mu(\lambda)+N^{\frac{-2r}{k}}\left(\int_{\lambda\geq N}|g(\lambda)|^2 d\mu(\lambda)\right)^{\frac{2k-1}{2k}}$$
$$\times\|\Delta_h^k \mathrm{D}_{\alpha,\beta}^r f(x)\|^{1/k},$$

or

$$(1-O(Nh)^{-\alpha-\frac{1}{2}})\int_{\lambda\geq N}|g(\lambda)|^2 d\mu(\lambda)$$
$$=O(N^{\frac{-2r}{k}})\left(\int_{\lambda\geq N}|g(\lambda)|^2 d\mu(\lambda)\right)^{\frac{2k-1}{2k}}\|\Delta_h^k \mathrm{D}_{\alpha,\beta}^r f(x)\|^{1/k}.$$

Setting $h=\frac{c}{N}$, in the last inequality and choosing $c>0$ such that $1-O(c^{-\alpha-\frac{1}{2}}\geq\frac{1}{2}$, we obtain

$$\int_{\lambda\geq N}|g(\lambda)|^2 d\mu(\lambda)=O(N^{\frac{-2r}{k}})\left(\int_{\lambda\geq N}|g(\lambda)|^2 d\mu(\lambda)\right)^{\frac{2k-1}{2k}}\psi^{\frac{1}{k}}((\frac{c}{N})^k),$$

we have

$$\int_{\lambda\geq N}|g(\lambda)|^2 d\mu(\lambda)=O(N^{-4r}\psi^2((\frac{c}{N})^k).$$

This completes the proof of theorem. □

Theorem 2. *Let $\psi(t)=t^\nu$, where $(\nu>0)$, then*

$$\sqrt{\int_{\lambda\geq N}|g(\lambda)|^2 d\mu(\lambda)}=O(N^{-2r-2k\nu})\Longleftrightarrow f\in \mathrm{W}_{2,\psi}^{r,k}(\mathrm{D}_{\alpha,\beta}),$$

$r=0,1,2....;\ k=1,2....;\ 0<\nu<2.$

Proof. Let $f \in \mathrm{W}^{r,k}_{2,\psi}(\mathrm{D}_{\alpha,\beta})$ and $\psi(t) = t^{\nu}$, by Theorem 1, we have

$$\left(\int_{\lambda \geq N} |g(\lambda)|^2 d\mu(\lambda)\right)^{1/2} = O(N^{-2r-k\nu}).$$

Suppose now that

$$\left(\int_{\lambda \geq N} |g(\lambda)|^2 d\mu(\lambda)\right)^{1/2} = O(N^{-2r-k\nu}).$$

It is easy to show, that there exists a function $f \in \mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$ such that $\mathrm{D}_{\alpha,\beta} f \in \mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$ and

$$\mathrm{D}_{\alpha,\beta} f(x) = \frac{(-1)^r}{2\pi} \int_0^{\infty} (\lambda^2 + \rho^2)^r \phi_{\lambda}^{(\alpha,\beta)}(x) d\mu(\lambda).$$

Hence, by the Plancherel equality, we have

$$\|\Delta_h^k \mathrm{D}^r_{\alpha,\beta} f(x)\|^2 = \int_0^{\infty} (\lambda^2 + \rho^2)^{2r} |1 - \phi_{\lambda}^{(\alpha,\beta)}(h)|^{2k} |g(\lambda)|^2 d\mu(\lambda).$$

Decompose this integral into two parts

$$\int_0^{\infty} = \int_{0<\lambda<N} + \int_{\lambda \geq N} = \mathrm{I}_1 + \mathrm{I}_2,$$

where $N = [N^{-1}]$, and estimate each of them.

We have

$$\begin{aligned}
\mathrm{I}_2 =& \int_{\lambda \geq N} (\lambda^2 + \rho^2)^{2r} |1 - \phi_{\lambda}^{(\alpha,\beta)}(h)|^{2k} |g(\lambda)|^2 d\mu(\lambda) \\
=& O\left(\int_{\lambda \geq N} (\lambda^2 + \rho^2)^{2r} |g(\lambda)|^2 d\mu(\lambda)\right) \\
=& O\left(\sum_{n=N}^{\infty} \int_n^{n+1} (\lambda^2 + \rho^2)^{2r} |g(\lambda)|^2 d\mu(\lambda)\right) \\
=& O\left(\sum_{n=N}^{\infty} ((n+1)^2 + \rho^2)^{2r} \int_n^{n+1} |g(\lambda)|^2 d\mu(\lambda)\right) \\
=& O\left(\sum_{n=N}^{\infty} n^{4r} \int_n^{n+1} |g(\lambda)|^2 d\mu(\lambda)\right)
\end{aligned}$$

$$
\begin{aligned}
&=O\left(\sum_{n=N}^{\infty} n^{4r}\int_{n}^{\infty}|g(\lambda)|^2d\mu(\lambda)-\sum_{n=N}^{\infty} n^{4r}\int_{n+1}^{\infty}|g(\lambda)|^2d\mu(\lambda)\right)\\
&=O\left(N^{4r}\int_{N}^{\infty}|g(\lambda)|^2d\mu(\lambda)+\sum_{n=N+1}^{\infty} n^{4r}\int_{n}^{\infty}|g(\lambda)|^2d\mu(\lambda)\right.\\
&\quad\left.-\sum_{n=N}^{\infty} n^{4r}\int_{n+1}^{\infty}|g(\lambda)|^2d\mu(\lambda)\right)\\
&=O\left(N^{4r}\int_{N}^{\infty}|g(\lambda)|^2d\mu(\lambda)+\sum_{n=N}^{\infty}((n+1)^{4r}-n^{4r})\int_{n}^{\infty}|g(\lambda)|^2d\mu(\lambda)\right)\\
&=O\left(N^{4r}\int_{N}^{\infty}|g(\lambda)|^2d\mu(\lambda)+\sum_{n=N}^{\infty} n^{4r-1}\int_{n}^{\infty}|g(\lambda)|^2d\mu(\lambda)\right)\\
&=O(N^{4r}N^{-4r-2k\nu}+\sum_{n=N}^{\infty} n^{4r-1}n^{-4r-2k\nu})\\
&=O(N^{-2k\nu})+O(N^{-2k\nu})=O(N^{-2k\nu})=O(h^{2k\nu}),
\end{aligned}
$$

i.e.

$$\mathrm{I}_2=O(h^{2k\nu}).$$

Now, we estimate I_1, by (2) in Lemma 2, and obtain

$$
\begin{aligned}
\mathrm{I}_1 &= \int_{0<\lambda<N}(\lambda^2+\rho^2)^{2r}|1-\phi_\lambda^{(\alpha,\beta)}(h)|^{2k}|g(\lambda)|^2d\mu(\lambda)\\
&= O(h^{4k})\int_{0<\lambda<N}(\lambda^2+\rho^2)^{2k+2r}|g(\lambda)|^2d\mu(\lambda)\\
&= O(h^{4k})\sum_{n=0}^{N}\int_{n}^{n+1}(\lambda^2+\rho^2)^{2k+2r}|g(\lambda)|^2d\mu(\lambda)\\
&= O(h^{4k})\sum_{n=0}^{N}(n+1)^{4r+4k}\int_{n}^{n+1}|g(\lambda)|^2d\mu(\lambda)\\
&= O(h^{4k})\sum_{n=0}^{N}(n+1)^{4r+4k}\left(\int_{n}^{\infty}|g(\lambda)|^2d\mu(\lambda)-\int_{n+1}^{\infty}|g(\lambda)|^2d\mu(\lambda)\right)\\
&= O(h^{4k})\left(\sum_{n=0}^{N}(n+1)^{4r+4k}\int_{n}^{\infty}|g(\lambda)|^2d\mu(\lambda)\right.\\
&\quad\left.-\sum_{n=0}^{N}(n+1)^{4r+4k}\int_{n+1}^{\infty}|g(\lambda)|^2d\mu(\lambda)\right)
\end{aligned}
$$

$$\begin{aligned}
&= O(h^{4k})\left(1+\sum_{n=0}^{N}((n+1)^{4r+4k}-n^{4r+4k})\int_{n}^{\infty}|g(\lambda)|^2d\mu(\lambda)\right)\\
&= O(h^{4k})\left(1+\sum_{n=1}^{N}n^{4r+4k-1}\int_{n}^{\infty}|g(\lambda)|^2d\mu(\lambda)\right)\\
&= O(h^{4k})\left(1+\sum_{n=1}^{N}n^{4r+4k-1}n^{-4r-2k\nu}\right)\\
&= O(h^{4k})\left(1+N^{4k-2k\nu}\right)=O(h^{2k\nu}),
\end{aligned}$$

i.e.

$$\mathrm{I}_1 = O(h^{2k\nu}).$$

Combining the estimates for I_1 and I_2 gives

$$\|\Delta_h^k \mathrm{D}_{\alpha,\beta}^r f(x)\| = O(h^{k\nu}),$$

which means that $f \in \mathrm{W}_{2,\psi}^{r,k}(\mathrm{D}_{\alpha,\beta})$.

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