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WEIGHTED RESIDUAL METHOD IN A SEMI INFINITE DOMAIN USING AN UN-PATITIONED METHODS

Razaq A. Oderinu[1 §], Y.A.S. Aregbesola[2]
[1,2]Ladoke Akintola University of Technology
PMB 4000, Ogbomoso, Oyo State, NIGERIA
[1]e-mail: adekola_razaq@yahoo.com
[2]e-mail: yaregbe@yahoo.com

Abstract: The weighted residual method is used to solve boundary value problems involving semi infinite domain by minimizing the domain within $[0-\infty]$ at once using the Gauss-Laguerre integration formula via the Galerkin and the moment methods.

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1. Introduction

Many physical problems in science and engineering, linear or nonlinear, such as heat flow and fluid flow in an infinite region are usually subject to boundary conditions at infinity. In Odejide and Aregbesola [6], the method of weighted residual was used to solve problems in semi infinite domains, where the domains were subdivided into subdomains, the finite parts of the subdomains were integrated using the Simpson $\frac{1}{3}$ rule and the remaining infinite parts were integrated using the shifted Laguerre method. In this paper the domain within 0 and ∞ is minimized directly without division using the Galerkin and the moment methods with direct Laguerre method of integration.

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[§]Correspondence author

2. Method of Solution

The idea of the method of weighted residuals is to seek an approximate solution, in form of a polynomial, to the differential equation of the form

$$L[u(x)] = f \quad \text{in the domain} \quad \Omega \tag{1}$$

$$B_\mu[u] = \Omega \quad \text{on} \quad \partial\Omega, \tag{2}$$

where $L[u]$ denotes a general differential operator (linear or non linear) involving spatial derivatives of dependent variable u. f is a known function of position, $B_\mu[u]$ represents the appropriate number of boundary conditions and Ω is the domain with a boundary $\partial\Omega$.

A trial function of the form

$$\phi = \phi_0 + \sum_{i=1}^{n} c_i \phi_i \tag{3}$$

is assumed, where c_i are constants to be determined which satisfy the given boundary condition (2). The trial function is chosen in such a way that it satisfies all the given boundary conditions including those at infinity.

Substitution of equation (3) into equation (1) gives the residual function $R(x)$. The idea is to minimize the residual function to be as small as possible. To minimize the residual along the whole domain, the following methods are used.

2.1. Galerkin Method

In the Galerkin method the weight functions ϕ_j, where $j = 1, 2, 3, ...n$, are used to multiply the residual $R(x)$, that is

$$\int_D \phi_j(x) R(x) dx = 0, \quad j = 1, 2, ...n. \tag{4}$$

The Gauss-Laguerre formula is now used to integrate each of the equations (4) to sets of simultaneous equations which can then be solved for the parameter c_i.

2.2. Moment Method

The weight functions $P_j(x)$ which are polynomials such that $P_j = x^j$, $j = 0, 1, 2, ...n$, are used to multiply the residual, that is

$$\int_D P_j(x) R(x) dx, \quad j = 1, 2, ...n. \tag{5}$$

The Gauss-Laguerre formula is now used to integrate each of the equations (5) to sets of simultaneous equations which can then be solved for the parameter c_i.

3. The Gauss-Laguerre Formula

To integrate within 0 to ∞, the procedure described below is used:

$$\int_0^\infty e^{-x} f(x) = \sum_{k=1}^{n} A_k f(x_k) + \frac{(n!)^2}{(2n)!} f^{(2n)}(\xi),$$

$$0 \leq \xi \leq \infty.$$

That is

$$\int_0^\infty e^{-x} f(x) dx \sim \sum A_k f(x_k),$$

where x_k are the zeros of the nth Laguerre polynomial

$$L_n(x) = e^x \frac{d^n}{dx^n}(e^{-x} x^n),$$

n is the number of points in consideration and

$$\frac{(n!)^2}{x_k (L_n'(x_k))^2}.$$

Table 1 shows the roots of the Laguerre polynomial and the corresponding weight function.

4. Numerical Experiments

Example 1. Consider the heat and mass problem obtained by Magyari and Keller [2]

$$f''' + ff' - 2f'^2 = 0,$$
$$f(0) = 0 \ , \ f'(0) = 1, \ f'(\infty) = 0.$$

Assuming the trial function

$$f = \sum_{i=0}^{n} c_i e^{\frac{-i\eta}{4}}$$

n	x_i	A_i
2	0.58578644	0.85355339
	3.41421356	0.14644661
4	0.32254769	0.60315410
	1.74576110	0.35741869
	4.53662030	0.03888791
	9.39507091	0.00053929
6	0.22284660	0.45896467
	1.18893210	0.41700083
	2.99273633	0.11337338
	5.77514357	0.1039920
	9.83746742	0.00026102
	15.98287398	0.00000090

Table 1

with $n = 8$, imposing the boundary condition $f(0) = 0$, we have

$$c_0 + c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8 = 0, \tag{6}$$

and for $f'(0) = 1$ we have

$$\frac{-1}{4}c_1 - \frac{1}{2}c_2 - \frac{3}{4}c_3 - c_4 - \frac{5}{4}c_5 - \frac{3}{2}c_6 - \frac{7}{4}c_7 - 2c_8 - 1 = 0. \tag{7}$$

The third condition is satisfied automatically. The residual

$$\begin{aligned}
R_1 =& \frac{-1}{64}c_1 e^{\frac{-1}{4}\eta} - \frac{1}{8}c_2 e^{\frac{-1}{2}\eta} - \frac{27}{64}c_3 e^{\frac{-3}{4}\eta} - c_4 e^{-\eta} - \frac{125}{64}c_5 e^{\frac{-5}{4}\eta} - \frac{27}{8}c_6 e^{\frac{-3}{2}\eta} \\
&- \frac{343}{64}c_7 e^{\frac{-7}{4}\eta} - 8c_8 e^{-2\eta} + (c_0 + c_1 e^{\frac{-1}{4}\eta} + c_2 e^{\frac{-1}{2}\eta} + c_3 e^{\frac{-3}{4}\eta} + c_4 e^{-\eta} \\
&+ c_5 e^{\frac{-5}{4}\eta} + c_6 e^{\frac{-3}{2}\eta} + c_7 e^{\frac{-7}{4}\eta} + c_8 e^{-2\eta})(\frac{1}{16}c_1 e^{\frac{-1}{4}\eta} + \frac{1}{4}c_2 e^{\frac{-1}{2}\eta} + \frac{9}{16}c_3 e^{\frac{-3}{4}\eta} \\
&+ c_4 e^{-\eta} + \frac{25}{16}c_5 e^{\frac{-5}{4}\eta} + \frac{9}{4}c_6 e^{\frac{-3}{2}\eta}\frac{49}{16}c_7 e^{\frac{-7}{4}\eta} + 4c_8 e^{-2\eta}) - 2(\frac{-1}{4}c_1 e^{\frac{-1}{4}\eta} \\
&- \frac{1}{2}c_2 e^{\frac{-1}{2}\eta} - \frac{3}{4}c_3 e^{\frac{-3}{4}\eta} - c_4 e^{-\eta} - \frac{5}{4}c_5 e^{\frac{-5}{4}\eta} - \frac{3}{2}c_6 e^{\frac{-3}{2}\eta} \\
&- \frac{7}{4}c_7 e^{\frac{-7}{4}\eta} - 2c_8 e^{-2\eta})^2,
\end{aligned}$$

using the moment method, that is integrating $R_1 * x^j, j = 2, 3, 4, 5, 6, 7, 8$, gives seven nonlinear equations which are solved alongside equations (6) and (7) to obtain the constants c_i as

$$c_0 = 0.9054214510, \quad c_1 = -0.0003373911855, \quad c_2 = 0.009859844685,$$

$$c_3 = -0.1689576664, \quad c_4 = -0.7902632581, \quad c_5 = 0.5031173313,$$
$$c_6 = -0.8123480444, \quad c_7 = 0.8210165218, \quad c_8 = -0.4675087886,$$

and

$$f(\infty) = 0.9054214510, \quad f''(0) = -1.280192397,$$

which compares favourably with the exact solution $f_{exact}(\infty) = 0.905639$ and $f''_{exact}(0) = -1.281808$ which are the universal constants of the problem in consideration according to the reference.

Example 2. Consider the fluid problem obtained by Mostafa [1]

$$f''' + ff'' - (D + Re)f' - (1 + \alpha)f'^2 = 0,$$
$$f(0) = 0, f'(0) = 1, f'(\infty) = 0,$$

whose exact solution is $\frac{1}{\sqrt{1+D+Re}}(1 - e^{-\eta\sqrt{1+D+Re}})$ when $\alpha = 0, Re = 0, D = 1$. Then we have

$$f''' + ff' - f' - f'^2 = 0.$$

Using the procedure discussed in Example 1, we have

$$f''(0) = -1.414217675$$

which compares favourably with the exact solution

$$f''(0) = -1.414213568\,.$$

Example 3. Consider also the Blasius equation governed by

$$f''' + \tfrac{1}{2}ff' = 0,$$
$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1.$$

Assuming the trial function

$$f = b\eta + \sum_{i=0}^{n} c_i e^{\frac{-i\eta}{4}}$$

with $n = 8$, imposing the boundary conditions, for $f(0) = 0$ we have

$$c_0 + c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8 = 0, \tag{8}$$

and for $f'(0) = 0$ we have

$$b - \frac{1}{4}c_1 - \frac{1}{2}c_2 - \frac{3}{4}c_3 - c_4 - \frac{3}{2}c_6 + \frac{7}{4}c_7 - 2c_8 = 0, \tag{9}$$

and for $f'(\infty) = 1$, we have

$$b = 1, \tag{10}$$

the residual is

$$\begin{aligned} R = & \frac{-1}{64}c_1 e^{\frac{-1}{4}\eta} - \frac{1}{8}c_2 e^{\frac{-1}{2}\eta} - \frac{27}{64}c_3 e^{\frac{-3}{4}\eta} - c_4 e{-}\eta \\ & - \frac{125}{64}c_5 e^{\frac{-5}{4}\eta} - \frac{27}{8}c_6 e^{\frac{-3}{2}\eta} - \frac{343}{64}c_7 e^{\frac{-7}{4}\eta} - 8c_8 e^{-2\eta} \\ & + (b\eta + c_0 + c_1 e^{\frac{-1}{4}\eta} + c_2 e^{\frac{-1}{2}\eta} + c_3 e^{\frac{-3}{4}\eta} + c_4 e^{-\eta} + c_5 e^{\frac{-5}{4}\eta} + c_6 e^{\frac{-3}{2}\eta} \\ & + c_7 e^{\frac{-7}{4}\eta} + c_8 e^{-2\eta})(\frac{1}{16}c_1 e^{\frac{-1}{4}\eta} + \frac{1}{4}c_2 e^{\frac{-1}{2}\eta} + \frac{9}{16}c_3 e^{\frac{-3}{4}\eta} \\ & + c_4 e^{-\eta} + \frac{25}{16}c_5 e^{\frac{-5}{4}\eta} + \frac{9}{16}c_6 e^{\frac{-3}{2}\eta} + \frac{49}{16}c_7 e^{\frac{-7}{4}\eta} + 4c_8 e^{-2\eta}. \end{aligned}$$

Using the Galerkin method, this residual is multiplied by $e^{\frac{-j}{4}x}$, $j = 0, 1, 2, 3, 4, 5, 6$, integrated over the domain to give seven nonlinear equations which are solved alongside equations (8), (9) and (10) to obtain

$$b = 1.00, \quad c_0 = -1.694694368, \quad c_1 = -0.1261928578,$$

$$c_2 = 2.409555114, \quad c_3 = -17.28681901, \quad c_4 = 48.36037065,$$

$$c_5 = -45.67439870, \quad c_6 = 6.358623767,$$

$$c_7 = 13.28213529, \quad c_8 = -5.628579882$$

and

$$f''(0) = 0.33391199,$$

which is a very good result also compare with Howarth solution,

$$f''_{Howarth} = 0.33206.$$

5. Conclusion

The method of the weighted residual was used to solve problems involving semi infinite domain with the use of the Gauss-Laguerre formula of integration over semi infinite domain on the interval without division. The method is simple, straightforward, and effective. The results are compared with the exact solution, where they are available and with the referenced solution, where they are not available.

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