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FIRST-PASSAGE TIME OF A STOCHASTIC INTEGRAL PROCESS THROUGH A LINEAR BOUNDARY

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Abstract: We study the probability distribution of the first-passage time of a stochastic integral process with deterministic integrand through a linear boundary, generalizing the well-known result regarding the Brownian motion.

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1. Introduction

Let B_t be a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, and let $T = \inf\{t > 0 : B_t \geq a + bt\}$ be the first-passage time of B_t through the linear boundary $S(t) = a + bt$. As it is well-known, if either $a \leq 0$ or $b \leq 0$, then T is finite with probability one, and it has the inverse Gaussian density, namely (see e.g. [8]):

$$\frac{d}{dt}P(T \leq t) = \frac{|a|}{t^{3/2}}\,\phi\left(\frac{a+bt}{\sqrt{t}}\right),\ t > 0, \tag{1}$$

where $\phi(y) = e^{-y^2/2}/\sqrt{2\pi}$; the probability distribution of T is furnished by the Bachelier-Levy formula (see e.g. [2]):

$$P(T \leq t) = 1 - \Phi(a/\sqrt{t} + b\sqrt{t}) + \exp(-2ab)\Phi(b\sqrt{t} - a/\sqrt{t}), \tag{2}$$

where $\Phi(x) = \int_{-\infty}^{x}\phi(y)dy$. In particular, for $b = 0$, one obtains

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$$P(T \le t) = 2\Psi\left(\frac{a}{\sqrt{t}}\right), \tag{3}$$

where $\Psi(x) = 1 - \Phi(x)$.

In this paper, we aim to generalize the above result to a stochastic integral process with deterministic integrand, such as

$$X(t) = \int_0^t \sigma(t) dB_t, \tag{4}$$

where $\sigma(t) > 0$ is a deterministic smooth function; so we focus on the first-passage time

$$\tau = \inf\{t > 0 : X(t) \ge a + bt\} \tag{5}$$

of $X(t)$ through the linear boundary $a + bt$; note that $X(t)$ turns out to be a time-changed Brownian motion, which is a very special case of diffusion process.

The first-passage time of a random processes through a moving barrier is very relevant in a variety of applications ranging from Physics, Biology and Engineering to mathematical finance (see e.g.[11], [12]), and in fact many papers have been devoted to study the first-passage time distribution of a diffusion through a curved boundary (see e.g. [5], [6], [9], [13] in general, and [4] about boundary-crossing of certain time-changed Brownian motions).

2. Main Results

Proposition 2.1. *Let*

$$X(t) = \int_0^t \sigma(s) dB_s \ , \tag{6}$$

where $\sigma(t) > 0$ is a deterministic continuous function. Let us suppose that positive constants A and B exist such that $A \le \sigma^2(t) \le B$, for any $t \ge 0$, and set $\rho(t) = \int_0^t \sigma^2(s)ds$. Then, if $a > 0$ and $b \le 0$ the first-passage time τ of $X(t)$ through the linear boundary $S(t) = a + bt$ is finite with probability one and the following bounds hold for the distribution function of τ :

$$\begin{aligned} 1 - \Phi(a/\sqrt{\rho(t)} + b\sqrt{\rho(t)}/B) + e^{-2ab/B}\Phi(b\sqrt{\rho(t)}/B - a/\sqrt{\rho(t)}) \\ \le P(\tau \le t) \\ \le 1 - \Phi(a/\sqrt{\rho(t)} + b\sqrt{\rho(t)}/A) + e^{-2ab/A}\Phi(b\sqrt{\rho(t)}/A - a/\sqrt{\rho(t)}). \end{aligned} \tag{7}$$

In particular, if $b = 0$ *one gets:*

$$P(\tau \leq t) = 2\Psi\left(\frac{a}{\sqrt{\rho(t)}}\right). \tag{8}$$

Proof. The function $\rho(t)$ is increasing and $\rho(0) = 0$; it represents the quadratic variation of $X(t)$, which is, for a general diffusion, an increasing stochastic process. From assumptions, we get $At \leq \rho(t) \leq Bt$, for any $t \geq 0$, and so $\rho(\infty) = \infty$; thus, a Brownian motion $\widetilde{B}_t$ exists, such that $X(t) = \widetilde{B}(\rho(t))$ (see e.g. [10]). Moreover, we have $\frac{s}{B} \leq \rho^{-1}(s) \leq \frac{s}{A}$. Then, $\tau = \inf\{t \geq 0 : \widetilde{B}(\rho(t)) - bt \geq a\}$ and $\widetilde{\tau} := \rho(\tau) = \inf\{s > 0 : \widetilde{B}_s - b\rho^{-1}(s) \geq a\}$. Note that, for any $b \leq 0$ and $s \geq 0$, it results:

$$\widetilde{B}_s - \frac{b}{B}\, s \leq \widetilde{B}_s - b\rho^{-1}(s) \leq \widetilde{B}_s - \frac{b}{A}\, s.$$

Now, if we consider the stopping times $\widetilde{T}_A := \inf\{s \geq 0 : \widetilde{B}_s - \frac{b}{A}\, s \geq a\}$ and $\widetilde{T}_B := \inf\{s \geq 0 : \widetilde{B}_s - \frac{b}{B}\, s \geq a\}$ (which are both finite with probability one, because they are first-passage times of Brownian motion with positive drift through a positive barrier), we obtain $\widetilde{T}_A \leq \widetilde{\tau} \leq \widetilde{T}_B$ and so

$$P(\widetilde{T}_B \leq t) \leq P(\widetilde{\tau} \leq t) \leq P(\widetilde{T}_A \leq t).$$

Since $P(\tau \leq t) = P(\widetilde{\tau} \leq \rho(t))$, we get:

$$P(\widetilde{T}_B \leq \rho(t)) \leq P(\tau \leq t) \leq P(\widetilde{T}_A \leq \rho(t)), \tag{9}$$

and therefore also τ is finite with probability one.

By computing the probabilities at the left and right-hand sides of (9) by means of the Bachelier-Levy formula (2), we easily obtain the bounds (7).

If $b = 0$, then $\tau = \inf\{t \geq 0 : \widetilde{B}(\rho(t)) \geq a\}$ and $\widetilde{\tau} = \rho(\tau) = \inf\{s > 0 : \widetilde{B}_s \geq a\}$. Since $P(\widetilde{\tau} \leq t) = 2\Psi\left(\frac{a}{\sqrt{t}}\right)$ (see (3)), we obtain

$$P(\tau \leq t) = P(\rho(\tau) \leq \rho(t)) = 2\Psi\left(\frac{a}{\sqrt{\rho(t)}}\right). \tag{10}$$

□

Remark 2.2. In the special case when $X(t)$ is Brownian motion, that is $\sigma(t) = 1 = A = B$ and $\rho(t) = t$, one obtains again the Bachelier-Levy formula, namely τ has the inverse Gaussian density (1).

Example. A simple integral process satisfying the assumption of Proposition 2.1 with A close to B is

$$X(t) = \int_0^t (1 + \epsilon \cos^2 s)\ \sigma\ dB_s,$$

where $\sigma > 0$, and $\epsilon > 0$ is small enough. We have

$$\rho(t) = \int_0^t (1 + \epsilon \cos^2 s)^2\ \sigma^2 ds,$$

so it results $At \leq \rho(t) \leq Bt$, with $A = \sigma^2$ and $B = (1+\epsilon)^2\sigma^2$. Since $B/A \approx 1$ for $\epsilon \approx 0$, the smaller is ϵ, the more precise the bounds (7) become.

Remark 2.3. The argument used in the proof of Proposition 2.1 allows to find, for instance, the first-passage distribution of the Ornstein-Uhlenbeck process over exponential-like boundaries. In fact, such a process $X(t)$ is the solution of the SDE:

$$dX(t) = -\mu X(t)dt + \sigma dB_t,\ X(0) = \eta,$$

where μ, σ are positive constants. The explicit solution is $X(t) = e^{-\mu t}Y(t)$, where $Y(t) = \eta + \int_0^t \sigma e^{\mu s} dB_s$. By using a time–change, we can write $Y(t) = \eta + \hat{B}_{\rho(t)}$, where $\hat{B}$ is Brownian motion and $\rho(t) = \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right)$. Then, $X(t) = e^{-\mu t}(\eta + \hat{B}(\rho(t)))$ and the first-passage time of $X(t)$ through the boundary $F(t) = (a + \eta - \frac{b\sigma^2}{2\mu})e^{-\mu t} + \frac{b\sigma^2}{2\mu}e^{\mu t} = e^{-\mu t}(a + \eta + b\rho(t))$ is

$$U = \inf\{t > 0 : X(t) \geq F(t)\} = \inf\{t > 0 : \hat{B}(\rho(t)) \geq a + b\rho(t)\}.$$

So, for $a > 0$ and $b \leq 0$, we get that $\rho(U) = \inf\{s > 0 : \hat{B}_s \geq a + bs\}$ has the inverse Gaussian density (1), and therefore from (2) it follows

$$P(U \leq t) = P(\rho(U) \leq \rho(t))$$
$$= 1 - \Phi(a/\sqrt{\rho(t)} + b\sqrt{\rho(t)}) + e^{-2ab}\Phi(b\sqrt{\rho(t)} - a/\sqrt{\rho(t)}).$$

The proof of Proposition 2.1 relies on the fact that positive constants A and B exist, such that $At \leq \rho(t) \leq Bt$. Of course, this could be not the case; if $\rho(t)$ is e.g. a concave increasing function, bounds to the distribution function of τ can be found by considering two families of continuous, piecewise-linear functions, g_t and h_t, which envelope $\rho(t)$ from above and below, prior to time t, namely:

$$h_t(s) \leq \rho(s) \leq g_t(s), \text{ for any s} \leq \text{t and } h_t(t) = g_t(t) = \rho(t),$$

where h_t and g_t have to be be chosen in suitable way (see [3]). Then, for $a > 0$ and $b < 0$, $S(u) =: a + b\rho^{-1}(u)$ is a concave decreasing function and we have $\widetilde{\tau} = \rho(\tau) = \inf\{u \geq 0 : \widetilde{B}_u \geq S(u)\}$; we obtain a polygonal approximation of S by enveloping it from above and below, prior to time $\rho(t)$, by means of $\tilde{g}_t(u) := a + bg_t^{-1}(u)$ and $\widetilde{h}_t(u) := a + bh_t^{-1}(u)$, with $\tilde{h}_t(u) \leq S(u) \leq \tilde{g}_t(u)$ for $u < \rho(t)$ and $\widetilde{h}_t(\rho(t)) = \widetilde{g}_t(\rho(t)) = S(\rho(t))$. Thus, the distribution of $\widetilde{\tau}$, and therefore that of τ, can be estimated by calculating the first-passage time distributions of $\widetilde{B}_s$ through the polygonal approximation of S, as done in Section 3.1 of [3].

Precisely, we choose the polygonal functions $\widetilde{g}_t(\cdot)$ enveloping S from above in such a way that they are formed by two straight lines which are, respectively, tangent to $S(u)$ in $u = 0$ and $u = \rho(t)$. These lines meet at the point having abscissa $v_t = \frac{S(\rho(t)) - S(0) - \rho(t)S'(\rho(t))}{S'(0) - S'(\rho(t))} < \rho(t)$. The polygonal functions $\widetilde{h}_t(\cdot)$ enveloping $S(u)$ from below are taken in such a way that they are formed by two straight lines which join the point $(0, S(0))$ with $(v_t, S(v_t))$ and $(v_t, S(v_t))$ with $(\rho(t), S(\rho(t)))$, that is

$$\widetilde{h}_t(u) = \begin{cases} a_1 + b_1 u, & 0 \leq u \leq v_t \\ a_2 + b_2 u, & v_t \leq u \leq \rho(t) \end{cases}, \tag{11}$$

with $a_1 = S(0) = a$, $b_1 = (S(v_t) - S(0))/v_t$, $a_2 = S(v_t) - v_t \frac{S(\rho(t)) - S(v_t)}{\rho(t) - v_t}$, $b_2 = \frac{S(\rho(t)) - S(v_t)}{\rho(t) - v_t}$;

$$\widetilde{g}_t(u) = \begin{cases} a'_1 + b'_1 u, & 0 \leq u \leq v_t \\ a'_2 + b'_2 u, & v_t \leq u \leq \rho(t) \end{cases}, \tag{12}$$

with $a'_1 = S(0) = a$, $b'_1 = S'(0) = \frac{b}{\rho'(0)}$, $a'_2 = S(\rho(t)) - \rho(t)(S'(\rho(t)) = a + bt - \frac{b\rho(t)}{\rho'(t)}$, $b'_2 = S'(\rho(t)) = \frac{b}{\rho'(t)}$.

Note that in [3] the definitions of the functions enveloping S from above and below have been interchanged, for error.

Now, we have:

$$\begin{aligned} P(\tau \leq t) &= P(\rho(\tau) \leq \rho(t)) = P(\widetilde{\tau} \leq \rho(t)) \\ &= P\left(\bigcup_{0 \leq s \leq \rho(t)} (\widetilde{B}_s \geq S(s)) \right). \end{aligned} \tag{13}$$

If $\widetilde{\tau}_h$ and $\widetilde{\tau}_g$ denote, respectively, the first-passage time of $\widetilde{B}$ through $\widetilde{h}_t$ and $\widetilde{g}_t$, we have $\widetilde{\tau}_h \le \widetilde{\tau} \le \widetilde{\tau}_g$ and so τ is finite with probability one, moreover:

$$P(\widetilde{\tau}_g \le \rho(t)) \le P(\widetilde{\tau} \le \rho(t)) \le P(\widetilde{\tau}_h \le \rho(t)). \tag{14}$$

The probabilities $P(\widetilde{\tau}_g \le \rho(t))$ and $P(\widetilde{\tau}_h \le \rho(t))$ can be exactly calculated by formula (3.7) in [3] which gives the first-passage time distribution of Brownian motion through a piecewise-linear boundary. Thus, we have obtained:

Proposition 2.4. *With the above notations, let us suppose that $\rho(t)$ is a concave function and set $k = \min(a_1 + b_1 v_t, a_2 + b_2 v_t)$ and $k' = \min(a_1' + b_1' v_t, a_2' + b_2' v_t)$. Moreover, set*

$$\Theta(x,y,t) = \Phi\left(\frac{x + y v_t - c}{\sqrt{\rho(t) - v_t}} + y\sqrt{\rho(t) - v_t}\right) \\ - \exp(-2(x + y v_t - c)y)\Phi\left(y\sqrt{\rho(t) - v_t} - \frac{x + y v_t - c}{\sqrt{\rho(t) - v_t}}\right).$$

Then, if $a > 0$ and $b \le 0$ the first-passage time τ of $X(t)$ through the linear boundary $S(t) = a + bt$ is finite with probability one and the following bounds hold for the distribution function of τ :

$$1 - \int_{-\infty}^{k'} \left(1 - \exp(-2a_1'(a_1' + b_1' v_t - c)/v_t)\right) \Theta(a_2', b_2', t) \frac{e^{-c^2/2v_t}}{\sqrt{2\pi v_t}}\, dc \le P(\tau \le t) \\ \le 1 - \int_{-\infty}^{k} \left(1 - \exp(-2a_1(a_1 + b_1 v_t - c)/v_t)\right) \Theta(a_2, b_2, t) \frac{e^{-c^2/2v_t}}{\sqrt{2\pi v_t}}\, dc. \tag{15}$$

If $\rho(t)$ is convex, one has to change the order, taking $\widetilde{h}_t$ as the enveloping curves from above and $\widetilde{g}_t$ those from below; then, analogous bounds for $P(\tau \le t)$ are obtained.

Remark 2.5. If one proceeds as done in [7] to find the first-passage time density of drifted Brownian motion through a constant barrier, by changing measure with Girsanov's theorem one can express the distribution of τ in a new probability measure Q as:

$$Q(\tau \le t) = \int_0^t Z_u P(T_a \in du), \tag{16}$$

where

$$Z_t = \exp\left(- \int_0^t \frac{b}{\sigma(s)}\, dB_s - \frac{1}{2}\int_0^t \left(\frac{b}{\sigma(s)}\right)^2 ds \right)$$

is a $\mathcal{F}_t-$martingale and T_a is the first-passage time of $X(t)$ through the barrier a, whose distribution is given by (10). Thus, under the assumptions of Proposition 2.1, we are able to estimate the integral in (16), and to obtain bounds for the distribution of τ under the measure Q, which are equivalent to those found in (7). In fact, the integral in (16) can be exactly computed only in the case when X is Brownian motion (i.e. $\sigma(t) \equiv 1$), this leading to the inverse Gaussian density for τ under the measure Q.

Remark 2.6. Some bounds to the first-passage time distribution through a linear boundary can be also found for a more general integral process $X(t)$ which is the solution of a SDE such as

$$dX(t) = \sigma(X(t))dB_t \ , X(0) = 0,$$

where $\sigma(x)$ is a regular enough function, for which the usual conditions for the existence and uniqueness of the solution hold. Because now the quadratic variation $\rho(t) = \int_0^t \sigma^2(X(s))ds$ is an increasing, but non-deterministic process, we assume that deterministic, increasing functions α and β exist, such that $\alpha(0) = \beta(0) = 0$ and $\alpha(t) \leq \rho(t) \leq \beta(t)$ for every $t > 0$. Consequently, also the pseudo-inverse of ρ is bounded between β^{-1} and α^{-1}, and one can substitute these estimates of ρ and its pseudo-inverse in the bounds already found for the first-passage time distribution through a linear boundary, when ρ is deterministic, in a similar way as done in [1] for calculating analogous quantities. Of course, in this way, we obtain weaker estimates for the first-passage time distribution; nevertheless, they can be useful in some circumstances.

3. Conclusions

In this short paper, we have considered a stochastic integral process such as $X(t) = \int_0^t \sigma(s)\ dB_s$, where B_t is Brownian motion and $\sigma(t)$ is continuous. In the case when σ is a deterministic function of t, we have found upper and lower bounds to the probability distribution of the first-passage time of $X(t)$ through the linear boundary $S(t) = a + bt$, with $a > 0,\ b \leq 0$, generalizing the well-known result regarding Brownian motion. If $\sigma(t)$ is state-dependent (i.e. $\sigma(t) = \sigma(X(t))$), we have discussed as the problem can be reduced to the previous case.

The first-passage time of a time-changed Brownian motion, such as $X(t)$, has interesting applications in mathematical finance, physics and many engineering problems.

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