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# BOUNDARY CONDITIONS FOR THE SOLUTION OF BOND PRICING EQUATION BY FINITE DIFFERENCES

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**Abstract:** A commonly available explicit solution of the bond pricing equation is derived by assuming that the logarithm of the value of a zero-coupon bond is a linear function of the short rate of interest. This solution of the bond pricing equation needs only the maturity condition and requires no boundary conditions even though the bond pricing equation is a backward parabolic partial differential equation. However, for a numerical solution by finite differences, the bond pricing equation requires specification of two appropriate boundary conditions. In the present paper we discuss some sets of boundary conditions for use with a finite difference solution of the bond pricing equation. Effectiveness of the derived boundary conditions is illustrated computationally for the Crank-Nicolson and generalized trapezoidal formula schemes for the numerical treatment of the bond pricing equation. It turns out that the condition of vanishing bond value when short rate increases unbounded is more difficult to implement using finite differences unless a more stable scheme is used; this is illustrated in the numerical experiments reported.

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## 1. Introduction

The bond pricing equation for the valuation of a zero-coupon bond is a backward parabolic partial differential equation. The final condition is specified by the maturity of the bond. For its complete specification, the bond pricing equation needs two boundary conditions. Since the range of the short rate is $(r_{\min}, \infty)$, a boundary condition for $r \to \infty$ is easily seen to be vanishing value of the bond there. The situation is more complex at $r_{\min}$ inasmuch as the short rate may or may not reach $r_{\min}$. If the short rate reaches $r_{\min}$, any valid boundary condition there can be taken while if the short rate does not reach $r_{\min}$, effectively no boundary condition can be imposed there. There can be a variety of solutions of the bond pricing equation for different specifications of boundary conditions. Not in all cases an explicit solution can be found, and even if one exists, it may be too involved for computational purposes. In that case then it might be more convenient to directly treat bond pricing equation numerically. For solution of the bond pricing equation numerically by finite differences we must impose a boundary condition at $r_{\min}$ whether short rate reaches it or not. Imposition of a boundary condition at $r = 0$ for the Cox-Ingersoll-Ross (CIR) model [5] and then the solution of the bond pricing equation by finite differences has been considered by Aquan-Assee [1], Ekstrom and Tysk [7], Ekstrom et al. [8], Kabanov et al. [10], Longstaff [12] and Mphaka and Taylor [13]. For pricing financial instruments by the finite difference method see, for example, Tavella and Randall [15].

A commonly available explicit solution of the bond pricing equation is derived by assuming that the logarithm of the value of a zero-coupon bond is a linear function of the short rate of interest. This solution of the bond pricing equation needs only the maturity condition and requires no boundary conditions even though the bond pricing equation is a backward parabolic partial differential equation. However, for a numerical solution by finite differences, the bond pricing equation requires specification of two appropriate boundary conditions. In the present paper we discuss some sets of boundary conditions for use with a finite difference solution of the bond pricing equation. Effectiveness of the derived boundary conditions is illustrated computationally for the Crank-Nicolson and generalized trapezoidal formula schemes for the numerical treatment of the bond pricing equation. It turns out that the condition of vanishing bond value when short rate increases unbounded is more difficult to implement using finite differences unless a more stable scheme is used; this is illustrated in the numerical experiments reported.

As in Chawla [2], following Wilmott et al. [17], we consider a four-parameter random walk model for the short term rate dynamics governed by:

$$dr = u(r,t)\,dt + w(r,t)\,dX, \tag{1.1}$$

where

$$w(r,t) = \sqrt{\alpha r - \beta}, \quad u(r,t) = (\eta - \gamma r) + \lambda(r,t)\,w(r,t).$$

Here, $\alpha$, $\beta$, $\gamma$ and $\eta$ are treated as constants and $dX$ is a Weiner process drawn from a normal distribution with mean zero and variance $dt$.

Let $B(r,t;T)$ (or, simply $B(r,t)$) denote the price of a bond at time $t$ with maturity $T$, $t < T$. The zero-coupon bond pricing equation, providing the value $B(r,t)$ of a bond at time $t < T$, is

$$\frac{\partial B}{\partial t} + \frac{1}{2}(\alpha r - \beta)\frac{\partial^2 B}{\partial r^2} + (\eta - \gamma r)\frac{\partial B}{\partial r} - rB = 0, \quad 0 < r < \infty, \; t \downarrow_0^T. \tag{1.2}$$

Note that $\lambda(r,t)$ does not appear in the bond pricing equation (1.2). The final condition is the payoff at maturity:

$$B(r,T) = Z,$$

and we need two appropriate boundary conditions to solve (1.2):

$$B(r = r_{\min}, t) = B_{\min}(t), \quad B(r = \infty, t) = B_{\infty}(t).$$

As noted above, a commonly available solution of the bond pricing equation is obtained in the form:

$$B(r,t;T) = Ze^{A(t;T) - rC(t;T)}, \tag{1.3}$$

satisfying the final conditions

$$A(T;T) = 0, \quad C(T;T) = 0,$$

with no reference to the boundary conditions even though the bond pricing equation is a parabolic equation. Set *time to maturity* of the zero-coupon bond $\tau = T - t$. A complete solution of the bond pricing equation in the form (1.3) has been given by Chawla [2]. For $\alpha > 0$, let

$$\psi = \sqrt{\gamma^2 + 2\alpha}, \quad a = \frac{-\gamma + \psi}{\alpha}, \quad b = \frac{\gamma + \psi}{\alpha},$$

then $C(t;T)$ is given by

$$C(t;T) = \frac{2}{\alpha}\frac{e^{\psi\tau}-1}{be^{\psi\tau}+a}, \tag{1.4}$$

and $A(t;T)$ is given by

$$\alpha A(t;T) = \beta C(t;T) + (\theta b - \beta)\tau - \frac{2\theta}{\alpha}\ln\left(\frac{be^{\psi\tau}+a}{b+a}\right), \tag{1.5}$$

where we have set $\theta = \alpha\eta - \beta\gamma$. Of particular interest is the Cox-Ingersoll-Ross (CIR) [5] extension of Vasicek model [16] which specifies that the instantaneous interest rate follows the stochastic differential equation:

$$dr = p(q-r)\,dt + \sigma\sqrt{r}dX. \tag{1.6}$$

The drift $p(q-r)$ is the same as in Vasicek model, ensuring mean reversion of interest rate towards the long run value $q$ with speed of adjustment governed by $p > 0$. However, in contrast with the Vasicek model, here the standard deviation function is $\sigma\sqrt{r}$ which avoids negative or zero interest rates for $\sigma^2 < 2pq$. In terms of our general short rate model (1.1), the CIR model is included as a special case for $\beta = 0$. Accordingly, solution of the bond pricing equation for the CIR model is given as in the following.

*The Case* $\beta = 0$, $\alpha$, $\gamma$, $\eta$ constant (Cox-Ingersoll-Ross model [5]):

For $\alpha > 0$, $C(t;T)$ is still given by (1.4). From (1.5) with $\beta = 0$ and $\theta = \alpha\eta$ we get

$$A(t;T) = \eta\left[b\tau - \frac{2}{\alpha}\ell n\left(\frac{be^{\psi\tau}+a}{b+a}\right)\right]. \tag{1.7}$$

## 2. Boundary Conditions

We consider here specification of appropriate boundary conditions for solution of the bond pricing equation (1.2) by finite differences.

**Boundary Condition at** $r = \infty$: First consider the boundary condition for $r \to \infty$. As noted by Karlin and Taylor [11], boundary at $r = \infty$ is inaccessible in finite time. Accordingly, no boundary condition can be imposed at $r = \infty$ for $0 < \tau < \infty$. However, to work with finite difference discretizations of the bond pricing equation, we need a boundary condition at $r = \infty$. For the purpose, behavior of the bond price $B(r,t)$ can be specified as defined implicitly by the bond pricing equation.

With the change $r = 1/R$, since

$$\frac{\partial B}{\partial r} = -R^2\frac{\partial B}{\partial R}, \ \frac{\partial^2 B}{\partial r^2} = 2R^3\frac{\partial B}{\partial R} + R^4\frac{\partial^2 B}{\partial R^2},$$

substituting into (1.2) and multiplying throughout by $R$, the bond pricing equation becomes

$$R\frac{\partial B}{\partial t} + \frac{1}{2}(\alpha - \beta R)R^3\left(2\frac{\partial B}{\partial R} + R\frac{\partial^2 B}{\partial R^2}\right) - (\eta R - \gamma)R^2\frac{\partial B}{\partial R} - B = 0. \quad (2.1)$$

If $\lim_{R\to 0} B(R,t)$ is finite (see discussion in the following), then taking limit $R \to 0$ from (2.1) it follows that $B(R=0,t) = 0$. Thus, the boundary condition at $r = \infty$ implied by the bond pricing equation (1.2) is

$$B(r=\infty,t) = 0, \ 0 < t < T. \quad (2.2)$$

It turns out that the boundary condition $B(r,t) \to 0$ for $r \to \infty$ is more difficult to implement using finite differences unless a more stable scheme is used; this is borne out by numerical experiments reported in Section 4.

**Boundary Conditions at Lower Bound** $r = \beta/\alpha$ **for** $\alpha > 0$: Throughout this paper we set

$$r_{\min} = \beta/\alpha, \ \delta = \frac{\eta - \gamma r_{\min}}{\alpha/2}.$$

The behavior at lower bound $r = r_{\min}$ is more complicated since this lower bound may or may not be reached by the short rate. We first consider behavior of solution of the bond pricing equation near this lower bound.

We seek a solution of the bond pricing equation (1.2) in the variable separable form

$$B(r,t) = U(r)V(t).$$

Substituting into the bond pricing equation we have

$$\frac{1}{V}\frac{\partial V}{\partial t} + \frac{1}{U}\left[\frac{1}{2}(\alpha r - \beta)\frac{\partial^2 U}{\partial r^2} + (\eta - \gamma r)\frac{\partial U}{\partial r} - rU\right] = 0.$$

It is clear that for a constant $k$ we must have

$$\frac{\partial V}{\partial t} = kV,$$

and

$$\frac{\alpha}{2}\left(r-r_{\min}\right)\frac{\partial^2 U}{\partial r^2}+\left(\eta-\gamma r\right)\frac{\partial U}{\partial r}-\left(r-k\right)U=0. \tag{2.3}$$

For the second order ordinary differential equation (2.3) for the determination of $U(r)$, $r = r_{\min}$ is a singular point. Moreover, it is easy to see that it is a regular singular point. We seek a power series solution in the form

$$U(r)=\sum_{n=0}^{\infty} a_n \left(r-r_{\min}\right)^{n+c}.$$

Since

$$\begin{aligned} p_0 &= \lim_{r\to r_{\min}}\left(\left(r-r_{\min}\right)\frac{\eta-\gamma r}{\frac{\alpha}{2}\left(r-r_{\min}\right)}\right)=\frac{\eta-\gamma r_{\min}}{\alpha/2}=\delta, \\ q_0 &= \lim_{r\to r_{\min}}\left(\left(r-r_{\min}\right)^2\frac{-\left(r-k\right)}{\frac{\alpha}{2}\left(r-r_{\min}\right)}\right)=0, \end{aligned}$$

it follows that the indicial equation (see, for example, Rainville et al. [14]) is

$$c\left(c-1\right)+\delta c=0,$$

giving the roots

$$c=0,\ c=1-\delta.$$

For a general solution of the bond pricing equation to be finite at $r = r_{\min}$ we must have $\delta < 1$, and the two linearly independent solutions are given by

$$U_1(r)=\sum_{n=0}^{\infty} a_n\left(r-r_{\min}\right)^{n},\ U_2(r)=\sum_{n=0}^{\infty} a_n\left(r-r_{\min}\right)^{n+1-\delta}.$$

If $\delta > 1$, a general solution is unbounded at $r = r_{\min}$. If $\delta = 1$, solution is still unbounded having a logarithmic singularity at $r = r_{\min}$. Finally, note that by the method of separation of variables, a solution of the bond pricing equation is given by

$$B(r,t)=e^{kt}U(r), \tag{2.4}$$

where $k$ is a constant independent of both $r$ and $t$.

It is now clear that for a general solution of the bond pricing equation to be finite at the lower bound for the short rate $r = r_{\min}$ which, in turn, means that the short rate can reach $r_{\min} = \beta/\alpha$, we must have

$$\delta < 1.$$

If $\delta \geq 1$, a general solution is unbounded at $r_{\min} = \beta/\alpha$; in both these cases the short rate can not reach $r_{\min} = \beta/\alpha$.

The above results are consistent with previously given criterion. No boundary condition is needed if the Fichera function, ignoring $\lambda$ since it does not appear in the bond pricing equation (1.2), $(\eta - \gamma r) - \frac{1}{2}\frac{\partial}{\partial r}(\alpha r - \beta)$ satisfies

$$\lim_{r \to \beta/\alpha} \left( (\eta - \gamma r) - \frac{1}{2}\frac{\partial}{\partial r}(\alpha r - \beta) \right) = \eta - \gamma \frac{\beta}{\alpha} - \frac{1}{2}\alpha \geq 0,$$

that is, if

$$\delta \geq 1.$$

This is also consistent with the usual Feller [9] condition which says that $r_{\min} = \beta/\alpha$ is inaccessible if $\eta \geq \beta\gamma/\alpha + \alpha/2$ and no boundary conditions can be imposed there.

When $\delta < 1$, $r_{\min} = \beta/\alpha$ is an attainable boundary and there can be many possible solutions to the bond pricing equation subject to the boundary conditions imposed and subject to the maturity condition. To completely specify a solution of the bond pricing equation, besides the boundary condition of vanishing of bond value for $r \to \infty$, we must impose an additional boundary condition at $r_{\min} = \beta/\alpha$. Different assumptions about the behavior of interest rate at $r_{\min} = \beta/\alpha$, resulting in different boundary conditions, will lead to different solutions of the bond pricing equation. Which assumption about boundary behavior at $r_{\min} = \beta/\alpha$ of the short rate of interest is more realistic is ultimately an empirical issue. Thus, in this sense, solving the bond pricing equation with appropriate boundary conditions is an open issue.

We next consider specifying boundary conditions at $r_{\min} = \beta/\alpha$ implied by the bond pricing equation for the case $\delta < 1$ when the short rate can reach $r_{\min}$.

**First Boundary Condition:** To obtain the first boundary condition, consider solution of the bond pricing equation by the separation of variables as in (2.4). At $r = r_{\min}$ this gives

$$B(r_{\min}, t) = e^{kt} U(r_{\min}) = e^{kt} U_1(r_{\min}) = a_0 e^{kt},$$

since $U_2(r_{\min}) = 0$. If this is to satisfy the maturity condition, then

$$B(r_{\min}, T) = a_0 e^{kT} = Z,$$

giving

$$B\left(r_{\min}, t\right) = Ze^{-k\tau}.$$

Since the value of a zero-coupon bond can not exceed its face value, therefore $k \geq 0$. Again, at $r = r_{\min}$, the bond pricing equation becomes

$$\frac{\partial B}{\partial t} + \left(\eta - \gamma r_{\min}\right)\frac{\partial B}{\partial r} - r_{\min} B = 0. \tag{2.5}$$

If the above boundary condition is to satisfy this equation, then $k = r_{\min}$ and

$$B\left(r_{\min}, t\right) = Ze^{-r_{\min}\tau}. \tag{2.6}$$

This means that the boundary value at $r_{\min}$ is the face value of the bond discounted at the rate $r_{\min}$. As $\tau$ increases, $B\left(r_{\min}, t\right)$ decreases implying short rate becomes greater than $r_{\min}$. Note that at $r = r_{\min}$, forward rate is $r_{\min}$. In particular, for the CIR model when $\eta < \alpha/2$, the boundary condition (2.6) becomes

$$B\left(r = 0, t\right) = Z.$$

This boundary condition would mean that the short rate is stuck at 0 if it reaches there, implying zero forward rates and zero expected return for each bond.

It may also be noted that for the first order partial differential equation (2.5), the characteristics are given by

$$\frac{dt}{1} = \frac{dr}{\eta - \gamma r_{\min}} = \frac{dB}{r_{\min}}.$$

One of the characteristics is

$$B\left(r_{\min}, t\right) = ce^{r_{\min} t},$$

for a constant $c$; if this is to satisfy the maturity condition then

$$B\left(r_{\min}, T\right) = ce^{r_{\min} T} = Z,$$

giving the boundary condition $B\left(r_{\min}, t\right) = Ze^{-r_{\min}\tau}$ as in (2.6).

We next obtain a solution of the bond pricing equation for the boundary condition (2.6) at $r = r_{\min}$. For this we set

$$\overline{r} = r - r_{\min},\ \overline{r}^{*} = \overline{r}\left(1 - \delta_{\tau,0}\right),\ \mu = \eta - \gamma r_{\min},$$

where $\delta_{\tau,0} = 1$ if $\tau = 0$ and 0 otherwise. Then, the bond pricing equation (1.2) can be written as

$$\frac{\partial B}{\partial t} + \frac{1}{2}\alpha \overline{r}^* \frac{\partial^2 B}{\partial \overline{r}^{*2}} + \left(\mu\left(1-\delta_{\tau,0}\right) - \gamma \overline{r}^*\right)\frac{\partial B}{\partial \overline{r}^*} - \left(\frac{\overline{r}^*}{1-\delta_{\tau,0}} + r_{\min}\right) B = 0. \tag{2.7}$$

We find a solution in the form

$$B\left(r,t\right) = Ze^{-r_{\min}\tau} U\left(\overline{r}^*\right).$$

Since $\frac{\partial}{\partial t} U\left(\overline{r}^*\right) = 0$, substituting in (2.7) we get

$$\frac{1}{2}\alpha^* \overline{r}^* \frac{\partial^2 U}{\partial \overline{r}^{*2}} + \left(\mu^* - \gamma^* \overline{r}^*\right)\frac{\partial U}{\partial \overline{r}^*} - \overline{r}^* U = 0, \tag{2.8}$$

where we have set $\left(\alpha^*, \mu^*, \gamma^*\right) = \left(\alpha, \mu, \gamma\right)\left(1-\delta_{\tau,0}\right)$. As before, we seek a power series solution in the form:

$$U\left(\overline{r}^*\right) = \sum_{n=0}^{\infty} a_n \left(\overline{r}^*\right)^{n+c}.$$

The indicial equation is the same as before, since $\delta = \frac{\mu}{\alpha/2}$, giving

$$c = 0, \;\; c = 1 - \delta.$$

We find one solution $U_1\left(\overline{r}^*\right)$ corresponding to $c = 0$, then the general solution can be found by the method of reduction of order. For the purpose, let

$$U\left(\overline{r}^*\right) = U_1\left(\overline{r}^*\right) v\left(\overline{r}^*\right),$$

and let

$$w\left(\overline{r}^*\right) = v'\left(\overline{r}^*\right).$$

Substituting in (2.8) and using the fact that $U_1\left(\overline{r}^*\right)$ is a solution of (2.8) and re-arranging, we obtain

$$w' + \left(\frac{2U_1'}{U1} + \frac{\delta}{\overline{r}^*} - \frac{\gamma}{\alpha/2}\right) w = 0.$$

Integrating factor for this first order ordinary differential equation is

$$IF = e^{\int\left(\frac{2U_1'}{U_1} + \frac{\delta}{\overline{r}^*} - \frac{\gamma}{\alpha/2}\right) d\overline{r}^*} = U_1\left(\overline{r}^*\right)^2 \left(\overline{r}^*\right)^{\delta} e^{-\left(\frac{\gamma}{\alpha/2}\right)\overline{r}^*},$$

therefore

$$w(\overline{r}^*) = C_1 \frac{e^{\left(\frac{\gamma}{\alpha/2}\right)\overline{r}^*}}{(\overline{r}^*)^{\delta} U_1(\overline{r}^*)^2}.$$

Then

$$v(\overline{r}^*) = C_2 + C_1 \int \frac{e^{\left(\frac{\gamma}{\alpha/2}\right)\overline{r}^*}}{(\overline{r}^*)^{\delta} U_1(\overline{r}^*)^2} d\overline{r}^*,$$

and finally we have

$$U(\overline{r}^*) = U_1(\overline{r}^*)\left[C_2 + C_1 \int \frac{e^{\left(\frac{\gamma}{\alpha/2}\right)\overline{r}^*}}{(\overline{r}^*)^{\delta} U_1(\overline{r}^*)^2} d\overline{r}^*\right],$$

for constants $C_1$ and $C_2$. Since we want this to satisfy the boundary conditions $U(\overline{r}=0) = 1$ and $U(\overline{r}=\infty) = 0$, clearly $C_2 = 0$, and with $U_1(0) = 1$, $C_1 = 1/K$, where we have set

$$K = \int_0^{\infty} \frac{e^{\left(\frac{\gamma}{\alpha/2}\right)s}}{s^{\delta} U_1(s)^2} ds,$$

and then

$$U(\overline{r}^*) = \frac{1}{K} U_1(\overline{r}^*) \int_{\overline{r}^*}^{\infty} \frac{e^{\left(\frac{\gamma}{\alpha/2}\right)s}}{s^{\delta} U_1(s)^2} ds. \tag{2.9}$$

Thus, the solution of the bond pricing equation (2.7) or, equivalently (1.2), is given by

$$B(r,t) = Ze^{-r_{\min}\tau} U(\overline{r}^*), \tag{2.10}$$

where $U(\overline{r}^*)$ is given by (2.9).

To find one solution of (2.8) corresponding to $c = 0$, let

$$U_1(\overline{r}^*) = \sum_{n=0}^{\infty} a_n (\overline{r}^*)^n.$$

Then, from (2.8) we obtain

$$\mu^* a_1 + \sum_{n=0}^{\infty}\left[\frac{\alpha^*}{2}(n+2)(n+1+\delta) a_{n+2} - \gamma^*(n+1) a_{n+1} - a_n\right](\overline{r}^*)^{n+1} = 0,$$

giving

$$\mu^* a_1 = 0,$$

and

$$\frac{\alpha^*}{2}(n+2)(n+1+\delta)a_{n+2} = \gamma^*(n+1)a_{n+1} + a_n, \ n = 0,1,2.... \quad (2.11)$$

First consider the case $\mu = 0$ which implies $\delta = 0$. In this case (2.11) reduces to

$$\frac{\alpha^*}{2}(n+2)(n+1)a_{n+2} = \gamma^*(n+1)a_{n+1} + a_n.$$

Taking $a_0 = 1$ and $a_1 = b$, we find $a_n = \frac{b^n}{n!}$. To verify it, we calculate

$$\begin{aligned} a_{n+2} &= \frac{1}{\frac{\alpha^*}{2}(n+2)(n+1)}\left[\gamma^*(n+1)\frac{b^{n+1}}{(n+1)!} + \frac{b^n}{n!}\right] \\ &= \frac{b^n}{(n+2)!}\left(\frac{\gamma^* b + 1}{\alpha^*/2}\right). \end{aligned}$$

Since

$$b^2 = \left(\frac{\gamma+\psi}{\alpha}\right)^2 = \frac{2}{\alpha^*}(1+\gamma^* b),$$

therefore $a_{n+2} = \frac{b^{n+2}}{(n+2)!}$. With this, for the case $\mu = 0$,

$$U_1(\overline{r}^*) = \sum_{n=0}^{\infty}\frac{b^n}{n!}(\overline{r}^*)^n = e^{b\overline{r}^*}.$$

From (2.9), since $\frac{\gamma}{\alpha/2} - 2b = -\frac{2\psi}{\alpha}$, we get

$$\begin{aligned} U(\overline{r}^*) &= \frac{1}{K}e^{b\overline{r}^*}\int_{\overline{r}^*}^{\infty} e^{-\left(\frac{2\psi}{\alpha}\right)s}ds \\ &= e^{\left(b-\frac{2\psi}{\alpha}\right)\overline{r}^*} = e^{-a\overline{r}^*}, \end{aligned}$$

and the solution of the bond pricing equation for this case is

$$B(r,t) = Ze^{-r_{\min}\tau}e^{-a\overline{r}^*}.$$

Note that, since $a > 0$, for $\tau > 0$ this solution satisfies the boundary condition $\lim_{r\to\infty} B(r,t) = 0$. Clearly, this also satisfies the other boundary condition at $r_{\min}$ and the final condition.

Now, consider the case $\mu \neq 0$. We set

$$\psi^* = \sqrt{\gamma^2 + 2\alpha(1+\delta)}, \ b^* = \frac{\gamma+\psi^*}{\alpha(1+\delta)}.$$

For the present assume that $a_1 \neq 0$ and let

$$a_n \geq \frac{b^{*n}}{n!},\ n = 0, 1, 2, \ldots.$$

Then from (2.11) we have

$$\frac{\alpha^*}{2}(n+2)(n+1+\delta)\, a_{n+2} \geq \frac{b^{*n}}{n!}(1+\gamma^* b^*).$$

Since

$$b^{*2} = \left(\frac{\gamma+\psi^*}{\alpha(1+\delta)}\right)^2 = \frac{2}{\alpha^*(1+\delta)}(1+\gamma^* b^*),$$

therefore,

$$(n+2)(n+1+\delta)\, a_{n+2} \geq \frac{b^{*n+2}}{n!}(1+\delta),$$

and then

$$a_{n+2} \geq \frac{b^{*n+2}}{(n+2)!}\frac{(n+1)(1+\delta)}{n+1+\delta} \geq \frac{b^{*n+2}}{(n+2)!}.$$

Thus,

$$U_1(\overline{r}^*) \geq e^{b^*\overline{r}^*} - b^*\overline{r}^*,$$

now accounting for the fact that $a_1 = 0$. Since $\overline{r}^* \leq s$, $U_1(\overline{r}^*) \leq U_1(s)$, from (2.9) we have

$$\begin{aligned} U(\overline{r}^*) &\leq \frac{1}{K(\overline{r}^*)^\delta}\int_{\overline{r}^*}^{\infty}\frac{e^{\left(\frac{\gamma}{\alpha/2}\right)s}}{U_1(s)}ds \\ &\leq \frac{1}{K(\overline{r}^*)^\delta}\int_{\overline{r}^*}^{\infty}\frac{e^{-\left(b^*-\frac{\gamma}{\alpha/2}\right)s}}{1-b^* s e^{-b^* s}}ds. \end{aligned}$$

Note that

$$b^* - \frac{\gamma}{\alpha/2} > 0 \ \text{ provided } \left(\frac{\gamma^2}{\alpha/2}\right)\delta < 1.$$

So, under this condition the integral in (2.9) converges and $\lim\limits_{\overline{r}^*\to\infty} U(\overline{r}^*) = 0$. Thus, in the case $\mu \neq 0$ also the bond price given by (2.10), for $\tau > 0$, satisfies the boundary condition $\lim\limits_{r\to\infty} B(r,t) = 0$. Clearly, this also satisfies the other boundary condition at $r_{\min}$ and the final condition.

Finally, we note that while there are four parameters in the model (1.1) for the short rate, the condition that the short rate can visit $r_{\min}$ is $\delta < 1$ and, for the boundary condition (2.6) to be met by a solution of the bond pricing equation, we have a further condition that $\left(\frac{\gamma^2}{\alpha/2}\right)\delta < 1$.

**Second Boundary Condition:** As noted before, a commonly available solution of the bond pricing equation is in non-separable variable form and assumes

$$B(r,t) = Ze^{A(t;T)-rC(t;T)}.$$

We refer to solution thus obtained as the *classical solution.* This tacitly assumes that $r_{\min} = 0$. Here, we start by assuming a slightly modified solution

$$B(r,t) = Ze^{A(t;T)-\overline{r}C(t;T)}. \tag{2.12}$$

Substituting the modified solution into (2.7) and rearranging gives

$$\left(\frac{\partial A}{\partial t} - \mu C - r_{\min}\right) + \overline{r}\left(-\frac{\partial C}{\partial t} + \frac{1}{2}\alpha C^2 + \gamma C - 1\right) = 0,$$

leading to the following two ordinary differential equations for the determination of $A(t;T)$ and $C(t;T)$:

$$\frac{\partial A}{\partial t} = \mu C + r_{\min}, \tag{2.13}$$

$$\frac{\partial C}{\partial t} = \frac{1}{2}\alpha C^2 + \gamma C - 1, \tag{2.14}$$

subject to final conditions $A(T;T) = 0$, $C(T;T) = 0$. Note that for $\overline{r} = 0$ only equation (2.13) is operative.

The solution of (2.14) can be taken as obtained in Chawla [2] and as given in (1.4). Now, integrating (2.13),

$$A(t;T) = \mu I + r_{\min}t + c,$$

where $c$ is constant of integration and we have set

$$I = \int C(t;T)\,dt.$$

As in Chawla [2]:

$$I = \frac{2}{\alpha}\left[\frac{\tau}{a} - \ln\left(be^{\psi\tau} + a\right)\right],$$

therefore

$$A(t;T) = \delta\left[\frac{\tau}{a} - \ln\left(be^{\psi\tau} + a\right)\right] + r_{\min}t + c.$$

From the final condition $A(T;T) = 0$ we get

$$c = \delta \ln(b + a) - r_{\min} T,$$

and thus

$$A(t;T) = \delta \left[ \frac{\tau}{a} - \ln\left( \frac{be^{\psi\tau} + a}{b + a} \right) \right] - r_{\min}\tau. \tag{2.15}$$

To obtain a boundary condition, clearly at $r = r_{\min}$, $\overline{r} = 0$, only equation (2.13) is operative and therefore its solution provides, from (2.12), the required boundary condition:

$$B(r_{\min}, t) = Ze^{A(t;T)}. \tag{2.16}$$

For $0 \leq \delta < 1$, $A(t;T)$ is negative. To see it, we first note that

$$\frac{\tau}{a} - \ln\left( \frac{be^{\psi\tau} + a}{b + a} \right) < \tau\left( \frac{1}{a} - \psi \right) - \ln\left( \frac{b}{b + a} \right).$$

Again, since $\frac{1}{a} = \frac{1}{2}(\gamma + \psi)$,

$$\frac{1}{a} - \psi = -\frac{1}{2}(-\gamma + \psi) = -\frac{\alpha a}{2},$$

then

$$\frac{\tau}{a} - \ln\left( \frac{be^{\psi\tau} + a}{b + a} \right) < -\frac{\alpha a}{2}\tau - \ln\left( \frac{b}{b + a} \right),$$

and therefore

$$A(t;T) < -\delta \left[ \frac{\alpha a}{2} + \ln\left( \frac{b}{b + a} \right) \right] - r_{\min}\tau.$$

It follows that $A(t;T) < 0$ for $0 \leq \delta < 1$. In view of this the boundary condition (2.16) implies strictly positive forward rates at $r = r_{\min}$ which means that the interest rate immediately becomes greater than $r_{\min}$ if it reaches $r_{\min}$. In passing, we also note that for $\tau \to \infty$, since

$$A(t;T) \,\tilde{}\, \tau \left[ \delta\left( \frac{1}{a} - \psi \right) - r_{\min} \right],$$

thus,

$$A(t;T) \,\tilde{}\, -\tau\left( \frac{\alpha a}{2}\delta + r_{\min} \right).$$

## 3. Finite Difference Methods

Since we are concerned with solution of the bond pricing equation by finite differences for specified boundary conditions, we consider a finite difference discretization of (1.2). As before, we set $t = T - \tau$ and let $B(r,t) = B(r, T-\tau) = u(r,\tau)$. Then, equation (1.2) can be written as a forward parabolic equation:

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}(\alpha r - \beta)\frac{\partial^2 u}{\partial r^2} + (\eta - \gamma r)\frac{\partial u}{\partial r} - ru, \quad 0 < r < \infty, \ 0 < \tau < T, \tag{3.1}$$

with the initial condition

$$u(r, \tau = 0) = Z,$$

and boundary conditions

$$u(r = r_{\min}, \tau) = u_{\min}(\tau), \quad u(r = \infty, \tau) = u_\infty(\tau).$$

Select an $r_\infty$. For a natural number $N$, consider the interest rate grid $r_i = r_{\min} + ih$, $i = 0, 1, ..., N+1$, where $h = (r_\infty - r_{\min}) / (N+1)$. Again, for a natural number $M$, consider the temporal grid $\tau_j = jk$, $j = 0, 1, ..., M$ where $k = T/M$. We set $u_{i,j} = u(r_i, \tau_j)$, etc.

With second order central difference discretizations for $\partial u / \partial r$ and $\partial^2 u / \partial r^2$, from (3.1) we obtain

$$\frac{\partial}{\partial \tau}u_i(\tau) = \frac{1}{2}(\alpha r_i - \beta)\left(\frac{u_{i+1}(\tau) - 2u_i(\tau) + u_{i-1}(\tau)}{h^2}\right) + (\eta - \gamma r_i)\left(\frac{u_{i+1}(\tau) - u_{i-1}(\tau)}{2h}\right) - r_i u_i(\tau), \quad i = 1, ..., N. \tag{3.2}$$

Let

$$\mathbf{u}(\tau) = \begin{bmatrix} u_1(\tau) \\ \cdot \\ \cdot \\ \cdot \\ u_N(\tau) \end{bmatrix}, \quad \mathbf{c}(\tau) = \begin{bmatrix} u_{\min}(\tau)(\alpha r_1 - \beta) \\ 0 \\ \cdot \\ 0 \\ u_\infty(\tau)(\alpha r_N - \beta) \end{bmatrix},$$

$$J = \begin{bmatrix} 2(\alpha r_1 - \beta) & -(\alpha r_1 - \beta) & & & \\ -(\alpha r_2 - \beta) & 2(\alpha r_2 - \beta) & -(\alpha r_2 - \beta) & & \\ & \cdot & \cdot & \cdot & \\ & & -(\alpha r_{N-1} - \beta) & 2(\alpha r_{N-1} - \beta) & -(\alpha r_{N-1} - \beta) \\ & & & -(\alpha r_N - \beta) & 2(\alpha r_N - \beta) \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & \eta - \gamma r_1 & & & & \\ -(\eta - \gamma r_2) & 0 & \eta - \gamma r_2 & & & \\ & \cdot & \cdot & & \cdot & \\ & & -(\eta - \gamma r_{N-1}) & & 0 & \eta - \gamma r_{N-1} \\ & & & & -(\eta - \gamma r_N) & 0 \end{bmatrix},$$

$$\mathbf{d}(\tau) = \begin{bmatrix} u_{\min}(\tau)(\eta - \gamma r_1) \\ 0 \\ \cdot \\ 0 \\ -u_\infty(\tau)(\eta - \gamma r_N) \end{bmatrix}, \quad R = \begin{bmatrix} r_1 & & & & \\ & r_2 & 0 & & \\ & & \cdot & & \\ & 0 & & r_{N-1} & \\ & & & & r_N \end{bmatrix},$$

then discretizations (3.2) can be written in matrix form as

$$\frac{\partial}{\partial \tau}\mathbf{u}(\tau) = \left(\frac{1}{2h^2}\mathbf{c}(\tau) - \frac{1}{2h}\mathbf{d}(\tau)\right) - \left(\frac{1}{2h^2}J + R - \frac{1}{2h}Q\right)\mathbf{u}(\tau). \tag{3.3}$$

We now set

$$\rho = \frac{k}{h^2}, \ \rho^* = \frac{k}{h},$$

and

$$A = \frac{\rho}{2}J + kR - \frac{\rho^*}{2}Q, \quad \mathbf{e}(\tau) = \frac{\rho}{2}\mathbf{c}(\tau) - \frac{\rho^*}{2}\mathbf{d}(\tau).$$

Then (3.3) can be rewritten as

$$\frac{\partial}{\partial \tau}\mathbf{u}(\tau) = \frac{1}{k}\left[\mathbf{e}(\tau) - A\mathbf{u}(\tau)\right]. \tag{3.4}$$

The Crank-Nicolson (C-N) scheme [6] is application of the classical trapezoidal formula for the time integration of (3.4), and is given by

$$\mathbf{u}_{j+1} = \mathbf{u}_j + \frac{k}{2}\left[\frac{1}{k}(\mathbf{e}_j - A\mathbf{u}_j) + \frac{1}{k}(\mathbf{e}_{j+1} - A\mathbf{u}_{j+1})\right],$$

leading to

$$\left(I + \frac{1}{2}A\right)\mathbf{u}_{j+1} = \left(I - \frac{1}{2}A\right)\mathbf{u}_j + \frac{1}{2}(\mathbf{e}_j + \mathbf{e}_{j+1}), \tag{3.5}$$

where $I$ is the identity matrix. Note that, at each time step of integration, the Crank-Nicolson scheme (3.5) requires the solution of a tridiagonal linear system.

Applying a generalized trapezoidal formula (GTF($\alpha_0$)) of Chawla et al. [4] (see also Chawla [3]) for the time integration of (3.4) we obtain the scheme

$$\left(I + \frac{1+\alpha_0}{2}A + \frac{\alpha_0}{2}A^2\right)\mathbf{u}_{j+1} = \left(I - \frac{1-\alpha_0}{2}A\right)\mathbf{u}_j$$

$$+\frac{1}{2}\left[\mathbf{e}_j + (I + \alpha_0 A)\,\mathbf{e}_{j+1}\right]. \tag{3.6}$$

Note that it requires the solution of a pentadiagonal linear system at each time step of integration. Note also that it includes the C-N scheme as a special case for $\alpha_0 = 0$.

Since the classical trapezoidal formula is A-stable but not L-stable, the C-N scheme is only A-stable. On the other hand, a GTF$(\alpha_0)$ is L-stable for all $\alpha_0 \in (0, 1]$ and only A-stable for $\alpha_0 = 0$ when it reduces to the C-N scheme. The L-stability of a GTF is desirable especially for integration with not-too-small time steps near $r_{\text{inf}}$ where there is a jump-down discontinuity in the interest rates curve and the C-N can experience wild oscillations near $r_{\text{inf}}$. This is borne out in numerical experiments reported in Section 4.

## 4. Numerical Illustrations

In the following we illustrate the use of finite differences for the valuation of bond prices for the suggested boundary conditions. In all the following examples, we fix $\alpha = 0.09$, $\beta = 0.0027$ and $\gamma = 0.55$ so that $r_{\min} = \beta/\alpha = 3\%$ and fix $N = 199$. We consider a zero-coupon bond of 10-year maturity with face value $Z = 100$, and consider its valuation when there remain 5 years to maturity for a range of values of interest rates above $r_{\min}$.

**Problem 1.** Besides the selection of a suitable boundary condition at $r_{\min}$, there are two crucial decisions to make before applying a finite difference method. The first concerns choice of $r_{\text{inf}}$ and the second concerns the choice of a finite difference method. The purpose of this first problem is to illustrate how these two decisions affect valuation of bond prices.

We first choose $\eta = 0.05$ so that $\delta = 0.74$ implying $r_{\min} = 3\%$ is attainable and we take the first suggested boundary condition $B(r_{\min}, t) = Ze^{-r_{\min}\tau}$. We select $r_{\text{inf}} = 50\%$ and $M = 100$ implying a time step of size 0.1. For the finite difference methods we consider C-N and GFT$(1/3)$. The choice of $\alpha_0 = 1/3$ for GTF is motivated by the fact that the bond pricing equation is linear and, as discussed in Chawla et al. [4], the choice of $\alpha_0 = 1/3$ in GTF$(\alpha_0)$ would enhance its order and accuracy besides being L-stable. The computed values of bond prices by these two finite difference methods are shown in Figure 1 alongwith the exact values. The C-N approximations suffer wild oscillations near $r_{\text{inf}}$. (These oscillations near $r_{\text{inf}}$ could be dismissed if we assume that, in practice, the interest rates may never be as high as $r_{\text{inf}}$.) These oscillations are due to the fact that the bond price curve has a jump-down discontinuity at $r_{\text{inf}}$ no matter whatever finite $r_{\text{inf}}$ we may choose. As noted earlier, C-N is based on the classical trapezoidal formula for time integration which is A-stable but not L-stable and is unable to cope with such jump discontinuities. On the other hand, GTF(1⁄3) gives satisfactory stable approximations being an L-stable time integration scheme. However, both these approximations are unacceptable since, in order to reach $r_{\text{inf}}$, both decrease too fast to reach chosen $r_{\text{inf}}$. Consequently, for the following examples we choose $r_{\text{inf}} = 1,000\%$, select GTF(1⁄3) for the finite difference method and take $M = 200$ so that the time step is of size 0.05.

**Problem 2.** We consider valuation of zero-coupon bond prices from the bond pricing equation with $\eta = 0.05$ implying $\delta < 1$ so that $r_{\text{inf}} = 3\%$ is attainable by the short rate. We take $r_{\text{inf}} = 1,000\%$ and $M = 200$ for a time step of 0.05. We again select the boundary condition $B(r_{\text{inf}}, t) = e^{-r_{\min}\tau}$. The bond values computed by GTF(1⁄3) are shown in Figure 2 alongwith the exact values. Clearly, the boundary condition works well and the GTF(1⁄3) provides both stable and accurate approximations for the bond prices.

**Problem 3.** The parameters are all the same as in Problem 2 so that $r_{\text{inf}} = 3\%$ is attainable. This time we take the second suggested boundary condition $B(r_{\text{inf}}, t) = Ze^{A(t;T)}$. The bond prices computed with GTF(1⁄3) are shown in Figure 3 alongwith the exact values. Again, this boundary condition also works well and GTF(1⁄3) provides stable and accurate approximations for the bond prices.

**Problem 4.** All the parameter values remain the same as in Problem 2 except that we now choose $\eta = 0.066$ so that $\delta = 1.1$ implying that now $r_{\text{inf}} = 3\%$ is unattainable by the short rate. In this case we can choose any boundary condition; however, we take the boundary condition $B(r_{\text{inf}}, t) = Ze^{-r_{\min}\tau}$. The bond prices computed by GTF(1⁄3) alongwith the exact values are shown in Figure 4. For this case of $r_{\text{inf}}$ being unattainable, the chosen boundary condition works well and again GTF(1⁄3) provides both stable and accurate approximations for the bond prices.

Finally, in Table 1 we show some representative bond prices for a few values of interest rates for Problems 2 and 3 with first and second boundary conditions, respectively, where $r_{\min} = 3\%$ is attainable, and for Problem 4 with first boundary condition where $r_{\min} = 3\%$ is unattainable.

| $r$ | Problem 2 | Problem 3 | Problem 4 |
|---|---|---|---|
| 3.00 | 86.0708 | 71.1769 | 86.0708 |
| 7.99 | 74.0842 | 65.8727 | 69.9904 |
| 12.97 | 66.6455 | 60.9733 | 62.0936 |

Table 1: Bond prices

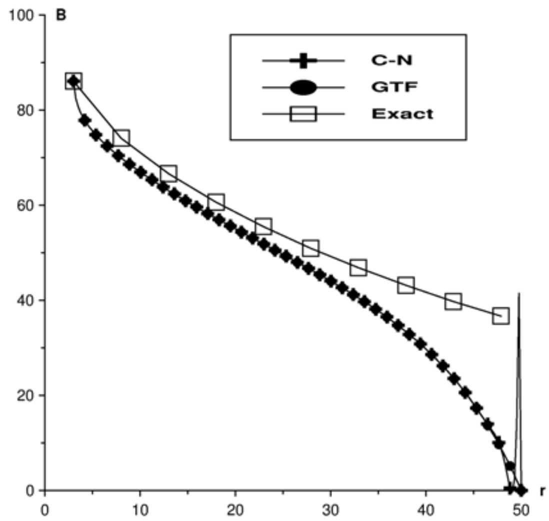


Figure 1: Problem 1

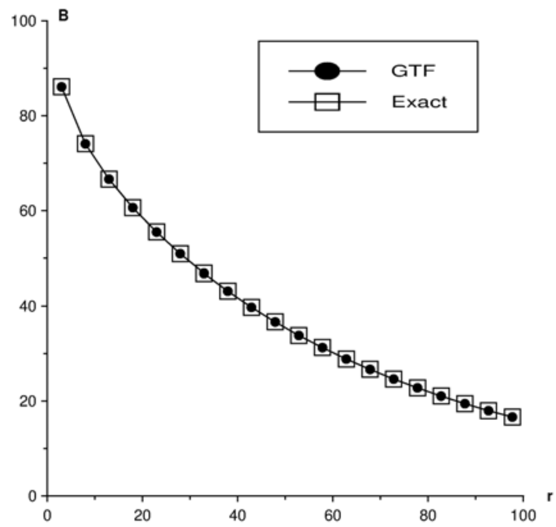


Figure 2: Problem 2

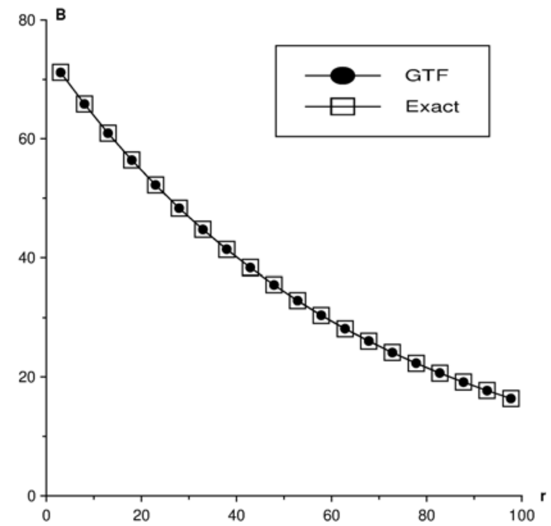


Figure 3: Problem 3

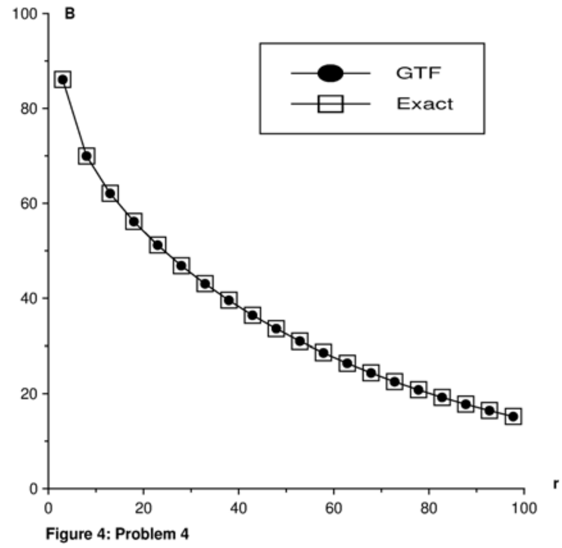


Figure 4: Problem 4

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