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SOME PROPERTIES AND APPLICATIONS OF GENERALIZED BERNOULLI POLYNOMIALS

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Abstract: We study some properties of the generalized Bernoulli polynomials related to the Riemann-Liouville fractional integrals. Using this, we generalize the asymptotic expansion of a function and the Euler-Maclaurin summation formula in terms of the generalized Bernoulli polynomials.

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1. Introduction

The generalized Bernoulli polynomials have important applications both in analytic theory of numbers and in classical and numerical analysis. Usually, the Bernoulli polynomials $B_n^{(\alpha)}(x)$ (real or complex) of order α are defined by the following generating function (López et al [5] and Nörlund [6])

$$\sum_{k=0}^{n}\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad |t| < 2\pi. \tag{1}$$

Their properties and explicit formulas are discussed by several authors, as e.g. Jordan[3], López et al [5], Nörlund [6], etc.

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The Bernoulli polynomials and numbers of order 1 appear in number theory, and in many mathematical expressions, such as the Taylor expansion in a neighborhood of the origin of the circular and hyperbolic tangent and cotangent functions (Gradshteyn et al [1]), the sums of powers of natural numbers (Gradshteyn et al [1], Jordan [3]), the Euler-Maclaurin summation formula (Hardy [2], Jordan [3], Nörlund [6]), etc. The purpose of this work is to generalize the asymptotic expansion of a function, Euler-Maclaurin summation formula, psi function, cotangent function through the Bernoulli polynomials of any positive order related to the Riemann-Liouville fractional integrals.

2. Properties of Generalized Bernoulli Polynomials

We begin by deriving some properties of the Bernoulli polynomials. Let us denote on $-\infty \leq 0 < x < \infty$, the Riemann-Liouville right-handed integral of f of fractional order $\alpha > 0$ is (Kilbas et al [4] and Podlubny[7])

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} f(u)du, \tag{2}$$

and the Grünwald-Letnikov right-handed fractional difference of g of order $\alpha > 0$ is (Kilbas et al [4] and Podlubny[7])

$$\Delta_x^\alpha g(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} g(x-k).$$

Theorem 1. *For some $\alpha > 0$ and $t > 0$*

$$\Delta_x^\alpha I_x^\alpha (e^{xt}) = e^{xt} \left(\frac{1-e^{-t}}{t} \right)^\alpha .$$

Proof. Replacing $f(x)$ by e^{xt} in (2) and after simplification, we find that

$$I_x^\alpha (e^{xt}) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} e^{ut} du = \frac{e^{xt}}{\Gamma(\alpha)} \int_0^x v^{\alpha-1} e^{-vt} dv.$$

We first prove the theorem, if α is an integer (assume $\alpha = n$), so that

$$\Delta_x^n I_x^n (e^{xt}) = \Delta_x^n \frac{e^{xt}}{(n-1)!} \int_0^x v^{n-1} e^{-vt} dv.$$

Using integration by parts, we find that

$$= \Delta_x^n \frac{e^{xt}}{(n-1)!}\left(-\sum_{k=0}^{n-1}(n-1)^{(k)}x^{n-1-k}\frac{e^{-xt}}{t^{k+1}}+\frac{(n-1)!}{t^n}\right).$$

Applying the difference operator Δ_x^n and simplifying, we get

$$\Delta_x^n I^n(e^{xt}) = e^{xt}\left(\frac{1-e^{-t}}{t}\right)^n.$$

Now, we prove the theorem for any α $(\alpha > 0)$. Let us define

$$h(\alpha) = \Delta^\alpha \frac{e^{xt}}{\Gamma(\alpha)}\int_0^x v^{\alpha-1}e^{-vt}dv.$$

Also, we have $h(n) = e^{xt}\left(\frac{1-e^{-t}}{t}\right)^n$, and

$$\Delta_x^n h(0) = e^{xt}\left(\frac{1-e^{-t}}{t}-1\right)^n.$$

Then, expanding $h(\alpha)$ by Newton's forward difference formula (Jordan [3]) and using binomial formula, we obtain the theorem. □

Theorem 2. *For $\alpha > 0$ and $n \in W$,*

$$\begin{aligned} \Delta_\alpha^\alpha I_x^\alpha B_n^{(\alpha)}(x) &= 1 \quad \textit{if } n = 0, \\ &= 0 \quad \textit{if } n > 0. \end{aligned} \tag{3}$$

Proof. Applying the Riemann-Liouville right-handed fractional integral of order α and the Grünwald-Letnikov right-handed fractional difference of order α on both sides of (1), we have

$$\left(\frac{t}{e^t-1}\right)^\alpha \Delta_x^\alpha I_x^\alpha(e^{xt}) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\Delta_x^\alpha I_x^\alpha B_n^{(\alpha)}(x).$$

By using Theorem 1 and simplifying, we get

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\Delta_x^\alpha I_x^\alpha B_n^{(\alpha)}(x) = e^{(x-\alpha)t}.$$

Now, putting $x = \alpha$, we find that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \Delta_{\alpha}^{\alpha} I_x^{\alpha} B_n^{(\alpha)}(x) = 1.$$

Comparing the coefficients of the powers of t on LHS and RHS of the above equation, we obtain (3). $\square$

Similarly, let us denote on $-\infty < x < 0 \leq \infty$, the Riemann-Liouville left handed integral of f of fractional order $\alpha > 0$ (Kilbas et al [4] and Podlubny [7]),

$${}_xI^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^0 (u - x)^{\alpha - 1} f(u) du,$$

and the Grünwald-Letnikov left-handed fractional difference of order α (Kilbas et al [4] and Podlubny[7]) as follo:

$${}_x\Delta^{\alpha} g(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} g(x + k).$$

Then, for some $\alpha > 0$ and $t > 0$,

$${}_x\Delta^{\alpha}{}_xI^{\alpha}(e^{-xt}) = e^{-xt} \left(\frac{1 - e^{-t}}{t} \right)^{\alpha}.$$

This gives, for $\alpha > 0$ and $n \in W$

$$\begin{aligned} {}_0\Delta^{\alpha}{}_xI^{\alpha} B_n^{(\alpha)}(x) &= 1 \quad \text{if } n = 0, \\ &= 0 \quad \text{if } n > 0. \end{aligned} \tag{4}$$

The result (4) is obtained in a similar way as (3). This clears that for any Bernoulli polynomial of order α ($\alpha > 0$), the right-handed fractional difference of Riemann-Liouville right-handed fractional integral at $x = \alpha$ is same as the left-handed fractional difference of the Riemann-Liouville left-handed fractional integral at $x = 0$. Hence, we use the right-handed operators $\Delta_{\alpha}^{\alpha} I_x^{\alpha}$ in the following section.

3. Asymptotic Expansion of a Function

In this section, we derive an asymptotic expansion of a function through the generalized Bernoulli polynomials.

Theorem 3. *Let $f(x)$ be a function with continuous $m+1$ derivatives and defined on the interval $[0,\infty)$, then*

$$f(x) = \sum_{k=0}^{m} \frac{B_k^{(\alpha)}(x)}{k!} \Delta_\alpha^\alpha I_t^\alpha f^{(k)}(t) + R_{m+1}(x), \tag{5}$$

where

$$R_{m+1}(x) = \frac{1}{m!} \Delta_\alpha^\alpha I_t^\alpha \int_t^x B_m^{(\alpha)}(x+t-u) f^{(m+1)}(u) du.$$

Proof. Denote $R_{k+1}(x)$, for $0 \le k \le m$ and $\alpha > 0$ as follows

$$R_{k+1}(x) = \frac{1}{k!} \Delta_\alpha^\alpha I_t^\alpha \int_t^x B_k^{(\alpha)}(x+t-u) f^{(k+1)}(u) du.$$

Now using integration by parts, we find that

$$R_{k+1}(x) = \frac{1}{k!} \Delta_\alpha^\alpha I_t^\alpha \Big(B_k^{(\alpha)}(t) f^{(k)}(x) - B_k^{(\alpha)}(x) f^{(k)}(t) + k \int_t^x B_{k-1}^{(\alpha)}(x+t-u) f^{(k)}(u) du \Big).$$

Using Theorem 2, i.e. $\Delta_\alpha^\alpha I_t^\alpha B_k^{(\alpha)}(t) = 0$ and after simplification, we find

$$= -B_k^{(\alpha)}(x) \Delta_\alpha^\alpha I_t^\alpha f^{(k)}(t) + R_k(x).$$

Rearranging above equation, we have

$$R_{k+1}(x) - R_k(x) = -\frac{1}{k!} B_k^{(\alpha)}(x) \Delta_\alpha^\alpha I_t^\alpha f^{(k)}(t).$$

Putting $k = 1, 2, 3, \dots, m$ and adding all set of equations

$$R_{m+1}(x) - R_1(x) = -\sum_{j=1}^{m} \frac{B_j^{(\alpha)}(x)}{j!} \Delta_\alpha^\alpha I_t^\alpha f^{(j)}(t).$$

But $R_1(x) = f(x) - \Delta_\alpha^\alpha I_t^\alpha f(t)$, substituting in above equation and after simplification, we obtain the theorem. □

Corollary 4. *Let $f(x)$ be a function with continuous $m+1$ derivatives and defined on the interval $[0,n]$*

$$f(x) = \sum_{k=0}^{m} \frac{B_k^{(n)}(x)}{k!} \Delta_n^n I_t^n f^{(k)}(t) + R_{m+1}(x),$$

where

$$R_{m+1}(x) = \frac{1}{m!}\Delta_n^n I_t^n \int_t^x B_m^{(n)}(x+t-u) f^{(m+1)}(u)du.$$

Proof. If we take $\alpha = n$ (where n is a positive integer) in Theorem 3, immediately we obtain the corollary. □

Remark 5. If $f(x)$ is a continuously differentiable function $[0,\infty)$, then letting $m \to \infty$ in Theorem 3, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{B_k^{(\alpha)}(x)}{k!} \Delta_\alpha^\alpha I_t^\alpha f^{(k)}(t).$$

If $f(x)$ is a continuously differentiable function on $[0,n]$, then letting $m \to \infty$ in Corollary 4, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{B_k^{(n)}(x)}{k!} \Delta_n^n I_t^n f^{(k)}(t).$$

The following identities for the generalized Bernoulli numbers are derived from the asymptotic expansion of $f(x)$. Let us begin by expanding $B_p^{(\alpha)}(x)$ using (5), then

$$B_p^{(\alpha)}(x) = \frac{p!}{m!(p-m-1)!}\Delta_\alpha^\alpha I_t^\alpha \int_t^x B_m^{(\alpha)}(x+t-u) B_{p-m-1}^{(\alpha)}(u)du.$$

Setting $x = 0$ and expanding Bernoulli polynomials in terms of Bernoulli numbers, we find that

$$B_p^{(\alpha)} = \frac{p!}{m!(p-m-1)!} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} B_{m-i}^{(\alpha)} B_{p-m-1-j}^{(\alpha)} \binom{m}{i}\binom{p-m-1}{j}$$
$$\times \Delta_\alpha^\alpha I_t^\alpha \int_t^0 (t-u)^i u^j du.$$

Since $\int_t^0 (t-u)^i u^j du = -\frac{j!i!}{(i+j+1)!} t^{i+j+1}$ and simplifying, we find that

$$B_p^{(\alpha)} = -p! \sum_{i=0}^{m} \sum_{j=0}^{p-m-1} c_{i,j}^{(p)} B_{m-i}^{(\alpha)} B_{p-m-1-j}^{(\alpha)},$$

where

$$c_{i,j}^{(p)} = \frac{1}{(m-i)!(p-m-1-j)!} \frac{\Delta_\alpha^\alpha \left(t^{i+j+\alpha+1}\right)}{\Gamma(i+j+\alpha+2)}.$$

If p is any even (say $p = 2m$), then

$$B_{2m}^{(\alpha)} = -2m! \sum_{i=0}^{m} \sum_{j=0}^{m-1} c_{i,j}^{(2m)} B_{m-i}^{(\alpha)} B_{m-1-j}^{(\alpha)},$$

where

$$c_{i,j}^{(2m)} = \frac{1}{(m-i)!(m-1-j)!} \frac{\Delta_\alpha^\alpha \left(t^{i+j+\alpha+1}\right)}{\Gamma(i+j+\alpha+2)}.$$

If p is any odd (say $p = 2m+1$), then

$$B_{2m+1}^{(\alpha)} = -(2m+1)! \sum_{i=0}^{m} \sum_{j=0}^{m} c_{i,j}^{(2m+1)} B_{m-i}^{(\alpha)} B_{m-j}^{(\alpha)},$$

where

$$c_{i,j}^{(2m+1)} = \frac{1}{(m-i)!(m-j)!} \frac{\Delta_\alpha^\alpha \left(t^{i+j+\alpha+1}\right)}{\Gamma(i+j+\alpha+2)}.$$

In particular if $\alpha = 1$, then we have the following identities

$$B_{2m} = -2m! \sum_{i=0}^{m} \sum_{j=0}^{m-1} d_{i,j}^{(2m)} B_{m-i} B_{m-1-j},$$

where

$$d_{i,j}^{(2m)} = \frac{1}{(m-i)!(m-1-j)!} \frac{1}{(i+j+2)!}.$$

Also, $m \in N$

$$\sum_{i=0}^{m} \sum_{j=0}^{m} d_{i,j}^{(2m+1)} B_{m-i} B_{m-j} = 0,$$

where

$$d_{i,j}^{(2m+1)} = \frac{1}{(m-i)!(m-j)!} \frac{1}{(i+j+2)!}.$$

4. Generalized Euler-Maclaurin Summation Formula

In this section, we generalize the Euler-Maclaurin summation formula in terms of the Bernoulli polynomials of any positive order α.

Theorem 6. *Let $f(x)$ be a function with continuous $m+1$ derivatives and defined on the interval $[0,\infty)$,*

$$\sum_{i=0}^{p-1} f(x+i) = \sum_{k=0}^{m} \frac{B_k^{(\alpha)}(x)}{k!} \Delta_\alpha^{\alpha-1} I_t^\alpha \left(f^{(k)}(p+t) - f^{(k)}(t) \right) + E_x, \tag{6}$$

where

$$E_x = \frac{1}{m!} \Delta_\alpha^{\alpha-1} I_t^\alpha \int_t^x B_m^{(\alpha)}(x+t-u) \left(f^{(m+1)}(p+u) - f^{(m+1)}(u) \right) du.$$

Proof. The expansion of $f(x)$ on the interval $[i,\infty)$ $(i = 0,1,\ldots,p-1)$ is same as expanding $f(x+i), i = 0,1,2,\ldots,p-1$ on the interval $[0,\infty)$. So, we have

$$f(x+i) = \sum_{k=0}^{m} \frac{1}{k!} B_k^{(\alpha)}(x) \Delta_\alpha^\alpha I_t^\alpha f(t+i) + R_{m+1}(x,i),$$

where

$$R_{m+1}(x,i) = \frac{1}{m!} \Delta_\alpha^\alpha I_t^\alpha \int_t^x B_m^{(\alpha)}(x+t-u) f^{(m+1)}(u+i) du.$$

Taking summation on the both sides by $i = 0,1,2,\ldots,p-1$,

$$\begin{aligned}
\sum_{i=0}^{p-1} f(x+i) &= \sum_{i=0}^{p-1} \left(\sum_{k=0}^{m} \frac{1}{k!} B_k^{(\alpha)} \Delta_\alpha^\alpha I_t^\alpha f^{(k)}(t+i) + R_{m+1}(x,i) \right) \\
&= \sum_{k=0}^{m} \frac{1}{k!} B_k^{(\alpha)}(x) \sum_{i=0}^{p-1} \Delta_\alpha^\alpha I_t^\alpha f^{(k)}(t+i) + \sum_{i=0}^{p-1} R_{m+1}(x,i). \qquad (7)
\end{aligned}$$

But

$$\begin{aligned}
\sum_{i=0}^{p-1} \Delta_\alpha^\alpha I_t^\alpha f^{(k)}(t+i) &= \sum_{i=0}^{p-1} \Delta_\alpha^{\alpha-1} \left(\Delta I_t^\alpha f^{(k)}(t+i) \right) \\
&= \sum_{i=0}^{p-1} \Delta_\alpha^{\alpha-1} I_t^\alpha \left(f^{(k)}(t+i+1) - f^{(k)}(t+i) \right)
\end{aligned}$$

$$= \sum_{i=0}^{p-1} \Delta_\alpha^{\alpha-1} I_t^\alpha f^{(k)}(t+i+1) - \sum_{i=0}^{p-1} \Delta_\alpha^{\alpha-1} I_t^\alpha f^{(k)}(t+i).$$

Thus, we have

$$\sum_{i=0}^{p-1} \Delta_\alpha^\alpha I_t^\alpha f^{(k)}(t+i) = \Delta_\alpha^{\alpha-1} I_t^\alpha \left(f^{(k)}(p+t) - f^{(k)}(t) \right). \tag{8}$$

Similarly, we find

$$\sum_{i=0}^{p-1} R_{m+1}(x,i) = \frac{1}{m!} \sum_{i=0}^{p-1} \Delta_\alpha^\alpha I_t^\alpha \int_t^x B_m^{(\alpha)}(x+t-u) f^{(m+1)}(i+u) du,$$

that is

$$\sum_{i=0}^{p-1} R_{m+1}(x,i) = \frac{1}{m!} \Delta_\alpha^{\alpha-1} I_t^\alpha \int_t^x B_m^{(\alpha)}(x+t-u) \times \left(f^{(m+1)}(p+u) - f^{(m+1)}(u) \right) du. \tag{9}$$

Substituting (8) and (9) in (7), after simplification we obtain the theorem. $\square$

Corollary 7. *Let $f(x)$ be a function with continuous $m+1$ derivatives and defined on the interval $[0,n]$.*

$$\sum_{i=0}^{p-1} f(x+i) = \sum_{k=0}^{m} \frac{B_k^{(n)}(x)}{k!} \Delta_n^{n-1} I_t^n \left(f^{(k)}(p+t) - f^{(k)}(t) \right) + E_x, \tag{10}$$

where

$$E_x = \frac{1}{m!} \Delta_n^{n-1} I_t^n \int_t^x B_m^{(n)}(x+t-u) \left(f^{(m+1)}(p+u) - f^{(m+1)}(u) \right) du.$$

Proof. If we take $\alpha = n$ is a positive integer in Theorem 6, then we immediately obtain the corollary. $\square$

Remark 8. If f is continuously differentiable function $[0,\infty)$, then letting $m \to \infty$ in Corollary 7, we find that

$$\sum_{i=0}^{p-1} f(x+i) = \sum_{k=0}^{\infty} \frac{1}{k!} B_k^{(\alpha)}(x) \Delta_\alpha^{\alpha-1} I_t^\alpha \left(f^{(k)}(p+t) - f^{(k)}(t) \right).$$

If f is continuously differentiable function $[0,n]$, then $m\to\infty$ in Theorem 6, we find that

$$\sum_{i=0}^{p-1} f(x+i)=\sum_{k=0}^{n}\frac{1}{k!}B_k^{(n)}(x)\Delta_n^{n-1}I_t^n\left(f^{(k)}(p+t)-f^{(k)}(t)\right).$$

The following are few examples of applications of the Euler-Maclaurin summation formula in terms of the Bernoulli polynomials of any order $n\in N$.

1. The sum of powers of first p natural numbers is

$$\sum_{i=1}^{p} i^m=\sum_{i=0}^{m+n}\frac{m!}{i!(n+m-i)!}B_i^{(n)}\Delta_n^{n-1}\left((p+t)^{m+n-i}-t^{m+n-i}\right).$$

2. The logarithm of the gamma function, psi function and cotangent are generalized on $0<x<1$ as follows:

$$\begin{aligned}\log\Gamma(x+1)=\Delta_n^n I_t^n\log\Gamma(t+1)-B_1^{(n)}(x)\\ +\sum_{k=1}^{n}\frac{U_k}{k!}B_k^{(n)}(x)+\sum_{k=1}^{\infty}\frac{V_k}{(n+k)!}B_{n+k}^{(n)}(x),\end{aligned}$$

$$\psi(x+1)=-1+\sum_{k=1}^{n}\frac{U_k}{(k-1)!}B_{k-1}^{(n)}(x)+\sum_{k=1}^{\infty}\frac{V_k}{(n+k-1)!}B_{n+k-1}^{(n)}(x),$$

$$\begin{aligned}\pi\cot\pi x=\sum_{k=1}^{n}\frac{1}{x-k}-\sum_{k=1}^{n}\frac{U_k}{(k-1)!}B_{k-1}^{(n)}(x)\left((-1)^k+1\right)\\ -\sum_{k=1}^{\infty}\frac{V_k}{(n+k-1)!}B_{n+k-1}^{(n)}(x)\left((-1)^{n+k}+1\right),\end{aligned}$$

where

$$U_k=\frac{1}{(n-k)!}\Delta^{n-1}1^{n-k}\log 1\quad\text{and}\quad V_k=(-1)^{k-1}(k-1)!\Delta^{n-1}\frac{1}{1^k}.$$

Equations (5) and (6) can be rewritten by the Grünwald-Letnikov left-handed fractional difference of the Riemann-Liouville left-handed fractional integral, if we replace the operator $\Delta_\alpha^\alpha I_x^\alpha$ by ${}_0\Delta_x^\alpha I^\alpha$.

5. Conclusion

We have shown some properties and applications of the generalized Bernoulli polynomials related to the Riemann-Liouville fractional integrals. We have proved that the asymptotic expansion of a function and the Euler-Maclaurin summation formula can be generalized to the Bernoulli polynomials of any positive order α. Also, we have given the sum of powers of natural numbers and expansion of logarithms of the Γ, the ψ and the cot – functions in terms of the generalized Bernoulli polynomials.

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