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DIMENSION OF BIVARIATE SPLINE SPACE OF DEGREE SIX ON UNIFORM TYPE-2 TRIANGULATION

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Abstract: In this paper, by using the technique of the well-known Bernstein-Bézier representation, and enforcing some additional smoothness conditions, the dimension of bivariate spline space of degree six on uniform type-2 triangulation is determined. Compared with [1], this method reduces the dimension by $5mn$.

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Key Words: bivariate splines, uniform type-2 triangulation, Bernstein-Bézier, dimension

1. Introduction

Let $\Omega = [0, x_m] \otimes [0, y_n]$ be a rectangle. For $0 = x_0 < x_1 < \cdots < x_m$ and $0 = y_0 < y_1 < \cdots < y_n$, Ω is divided into mn small rectangle $\Omega_{ij} = [x_i, x_{i+1}] \otimes [y_j, y_{j+1}]$, $i = 0, 1, \cdots, m-1,\ j = 0, 1, \cdots, n-1$, by mesh lines,

$$x = x_i,\ i = 0, 1, \cdots, m-1, \text{ and } y = y_j,\ j = 0, 1, \cdots, n-1.$$

Let $h_i = x_i - x_{i-1},\ t_j = y_j - y_{j-1}$ for $i = 1, 2, \cdots, m,$ and $j = 1, 2, \cdots, n,$ If $h_i \equiv h,\ i = 1, 2, \cdots, m,$ and $t_j \equiv t,\ j = 1, 2, \cdots, n,$ and the triangulation generated

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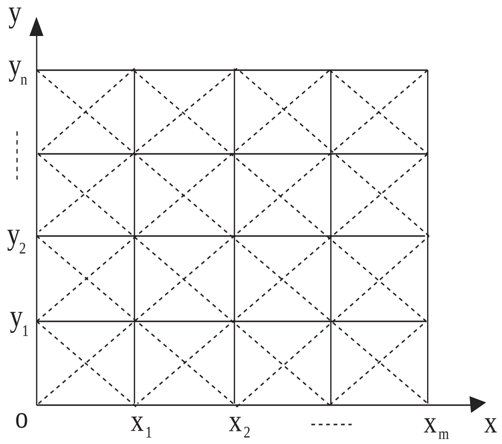

Figure 1: A uniform type-2 triangulation $\triangle_{mn}^{(2)}$)

by drawing all northeast and northwest diagonals in all small rectangles, then the triangulation is called uniform type-2 triangulation, and it is denoted by $\triangle_{mn}^{(2)}$ (Figure 1).

In this paper, we are interested in the spline function space of smoothness order 2 and degree 6:

$$S_6^2(\triangle_{mn}^{(2)}) = \left\{ s \in C^2(\Omega) : s|_T \in \mathcal{P}_6, \text{ for all triangles } T \text{ in } \triangle_{mn}^{(2)} \right\}, \tag{1}$$

where $\mathcal{P}_6$ denotes the space of bivariate polynomials of total degree being at most 6.

The dimension of $S_6^2(\triangle_{mn}^{(2)})$ can be determined by Chui and Wang's results [1]. In 1997, Lai and Schumaker [2] studied the C^2 supersline space of degree 6 on triangulated quadrangulation, this is a generalization of the results of [1]. In this paper, by enforcing some additional smoothness conditions, a minimal determining set of subspace of $S_6^2(\triangle_{mn}^{(2)})$ is constructed. It implies that the dimension of the space is reduced, compared to [1].

2. Preliminaries

Throughout the paper, we use the well-known Bernstein-Bézier technique of bivariate splines [3]. Let $T :=< v_1, v_2, v_2 >$ be a triangle in $\triangle$, where v_1, v_2 and v_3 are three vertices of T. Then every polynomial $s \in \mathcal{P}_d$ associated with

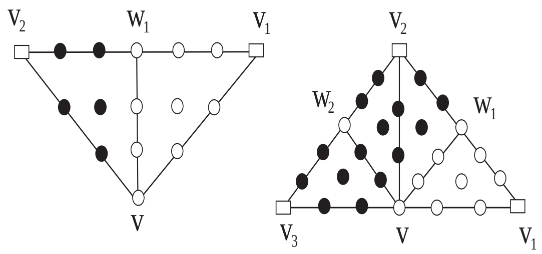

Figure 2: A minimal determining set for $S_3^2(star(v))(n = 2, n = 3)$ in Lemma 2.1.

T can be written as

$$s := \sum_{i+j+k=d} c_{ijk}^T \mathcal{B}_{ijk}^d, \tag{2}$$

where $\{\mathcal{B}_{ijk}^d\}_{i+j+k=d}$ is the Bernstein basis polynomials of degree d on the triangle T. The coefficients $\{c_{ijk}^T\}_{i+j+k=d}$ are called B-coefficients of s associated with domain points $\mathcal{D}_T := \{\xi_{ijk}^T := (iv_0 + jv_1 + kv_2)/d\}_{i+j+k=d}$. Let $\mathcal{D}_{d,\triangle}$ be the union of the sets of domain points associated with the triangles of $\triangle$.

Let $S_d^0(\triangle)$ be the space of continuous splines of degree d on the triangulation $\triangle$. A subset $\mathcal{M} \subseteq \mathcal{D}_{d,\triangle}$ is said to be a determining set for a spline space $S \subseteq S_d^0(\triangle)$ if $s \in S$ and $c_\xi = 0$ for all $\xi \in \mathcal{M}$, then $c_\eta = 0$ for all $\eta \in \mathcal{D}_{d,\triangle}$, *i.e.*, $s \equiv 0$. Moreover, $\mathcal{M}$ is called a minimal determining set for S if there is no smaller determining set. It is known that $\dim S = |\mathcal{M}|$, where $|\mathcal{M}|$ denotes the cardinality of $\mathcal{M}$.

Let $T :=< v_0, v_1, v_2 >$ and $\tilde{T} :=< v_3, v_2, v_1 >$ be two adjacent triangles, they share the common edge $e :=< v_1, v_2 >$. Let $s|_T$ and $s|_{\tilde{T}}$ to denote the restrictions of s on T and $\tilde{T}$ with B-coefficients c_{ijk} and $\tilde{c}_{ijk}$, respectively. Following [4], for any $0 \le n \le m \le d$, let $\tau_{e,m}^n$ be the linear functional defined on $S_d^0(\triangle)$ by

$$\tau_{e,m}^n s := \tilde{c}_{n,m-n,d-m} - \sum_{i+j+k=n} c_{i,j+d-m,k+m-n} \mathcal{B}_{ijk}^n(v_3), \tag{3}$$

then the condition that s is C^r smooth across the edge e is equivalent to

$$\tau_{e,m}^n s = 0, \qquad n \le m \le d, \quad 0 \le n \le r. \tag{4}$$

First we give there lemmas established in Lai and Schumaker [2].

Lemma 2.1. *Suppose that v is a boundary vertex of $\triangle_{mn}^{(2)}$ whit $2n-1$ edges attached, let the boundary vertices of $star(v)$ be $v, v_1, w_1, v_2, w_2, \cdots, w_{n-1}, v_n$ in counterclockwise order. Then the following set of $2n+7$ domain points is a MDS for $S_3^2(star(v))$:*
(1) ξ_{ijk}^{v,v_1,w_1}, $i+j+k=3$,
(2) $\xi_{030}^{v,v_i,w_i}, \xi_{003}^{v,v_i,w_i}$, $i=2,\cdots,n-1$,
(3) $\xi_{003}^{v,w_{n-1},v_n}$.
These points are marked with $\circ$ and $\Box$ in Fig. 2.

Lemma 2.2. *Suppose that v is an interior vertex of $\triangle_{mn}^{(2)}$ with 8 edges attached, let the boundary vertices of $star(v)$ be $v_1, w_1, v_2, w_2, v_3, w_3, v_4, w_4$ in counterclockwise order. Then the following set of 14 domain points is a MDS for $S_3^2(star(v))$:*
(1) ξ_{ijk}^{v,w_4,v_1}, $i+j+k=3$,
(2) $\xi_{030}^{v,v_i,w_i}, i=2,3,4$,
(3) ξ_{003}^{v,v_3,w_3}.
These points are marked with $\circ$ and $\Box$ in Fig. 3.

Lemma 2.3. *Let $star(v)$ consist 4 triangles surrounding a vertex v formed by two crossing lines. We denote the boundary vertices of star(v) by v_1, v_2, v_3, v_4. Let $T_l = < v, v_l, v_{l+1} >$ for $l=1,2,3,4$, where we identify $v_5 = v_1$. Then the following set of 49 domain points is a MDS for $S_6^2(star(v))$:*
(1) $\xi_{ijk}^{T_l}$, $i+j+k=6$, $j \geq 3$, $l=1,2,3,4$,
(2) $\xi_{222}^{T_l}$, $l=1,2,3,4$,
(3) $\xi_{420}^{T_l}$, $l=1,2,3,4$,
(4) $\xi_{600}^{T_1}$.
These points are marked with $\circ$, $\Box$, $\triangle$ and $*$ in Fig. 4.

Following Lemma 2.3, we can get a lemma by enforcing some additional smoothness conditions across the interior edges of $star(v)$.

Lemma 2.4. *Let $star(v)$ be a triangulation described in Lemma 2.3. Let $\hat{S}_6^2(star(v))$ be the subspace of $S_6^2(star(v))$ satisfying the following set of additional smoothness conditions:*

$$\tau_{e_i,4}^3 s = 0, \quad i=1,2,3,4 \text{ and } \tau_{e_1,6}^3 s = 0, \tag{5}$$

then $\dim \hat{S}_6^2(star(v)) = 44$, and the following set of 44 domain points is a MDS for $\hat{S}_6^2(star(v))$:
(1) $\xi_{ijk}^{T_l}$, $i+j+k=6$, $j \geq 3$, $l=1,2,3,4$,
(2) $\xi_{222}^{T_l}$, $l=1,2,3,4$,
These points are marked with $\circ$, $\Box$ and $\triangle$ in Fig. 4.

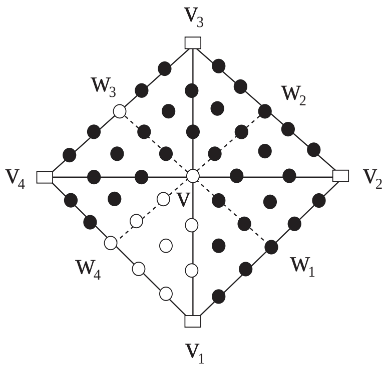

Figure 3: A minimal determining set for $S_3^2(star(v))$ in Lemma 2.2

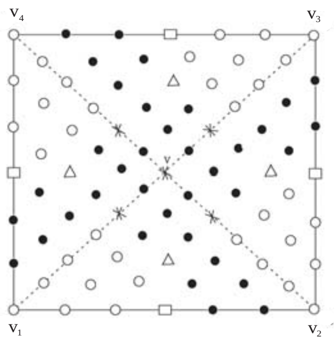

Figure 4: A minimal determining set for $S_3^2(star(v))$ in Lemma 2.3

Proof. We can prove this lemma easily by using Lemma 2.3 and conditions of (5). $\square$

3. Dimension of $\hat{S}_6^2(\triangle_{mn}^{(2)})$

Theorem 3.1. *Let $\triangle_{mn}^{(2)}$ be an uniform type-2 triangulation, let $\hat{S}_6^2(\triangle_{mn}^{(2)})$ be the set of all splines $s \in S_6^2(\triangle_{mn}^{(2)})$ such that s satisfies the additional smoothness conditions in each small rectangle of Ω as described in Lemma 2.4. Then*

$$\dim \hat{S}_6^2(\triangle_{mn}^{(2)}) = 14mn + 12(m+n) + 6.$$

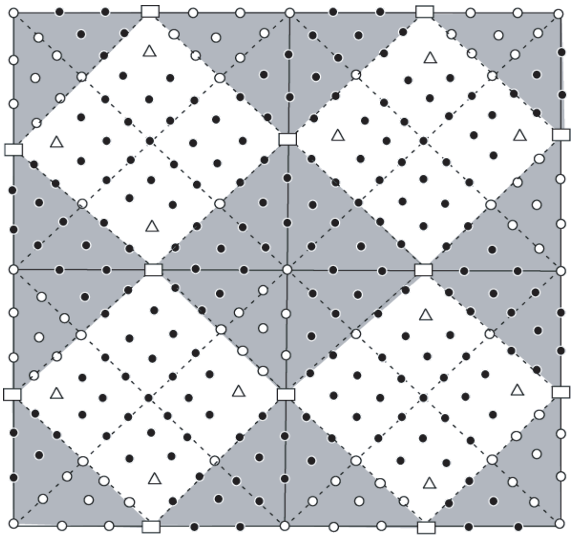

Figure 5: A minimal determining set for $\hat{S}^2_6(\triangle^{(2)}_{mn})$

Proof. We choose the following domain points set P to construct a determining set for $\hat{S}^2_6(\triangle^{(2)}_{mn})$:

(1) For each boundary vertices v of rectangle Ω, choose 9 or 10 points in the 3-disks around v as in Lemma 2.1 which are marked with $\circ$ in Fig. 2.

(2) For each interior vertices v of rectangle Ω, choose 10 points in the 3-disks around v as in Lemma 2.2 which are marked with $\circ$ in Fig. 3.

(3) For each edge of Ω, choose 2 points as in Lemma 2.4 which are marked with $\square$ and $\triangle$ in Fig. 4.

To show that P is a determining set, suppose all of the B-coefficients of $s \in \hat{S}^2_6(\triangle^{(2)}_{mn})$ in P to zero. Then by (1) and (2), using Lemma 2.1 and Lemma 2.2, all B-coefficients in the 3-disks around every vertex must be zero. Since the B-coefficients in (3) are zero, using the C^2 conditions across the edges we see that all B-coefficients of ξ_{222} on both sides of any edge of Ω are zero. Finally, Lemma 2.4 imply that s must vanish identically inside each small rectangle. So P is a determining set for $\hat{S}^2_6(\triangle^{(2)}_{mn})$, and $\dim \hat{S}^2_6(\triangle^{(2)}_{mn}) \le |P|$.

It is easy to see that

$$|P| = 10(m+1)(n+1) - 4 + 2(m+1)n + 2(n+1)m,$$

it is noted that $\dim S^2_6(\triangle^{(2))}_{mn} = 19mn + 12(m+n) + 6$ [1]. Our space $\hat{S}^2_6(\triangle^{(2)}_{mn})$ is the subspace of $S^2_6(\triangle^{(2))}_{mn})$ that satisfies $5mn$ additional special smoothness conditions. Thus,

$$19mn + 12(m+n) + 6 - 5mn \leq dim\hat{S}_6^2(\triangle_{mn}^{(2)}) \leq |P| = 14mn + 12(m+n) + 6,$$

and we conclude that $\dim \hat{S}_6^2(\triangle_{mn}^{(2)}) = 14mn + 12(m+n) + 6$, and P is a MDS for $\hat{S}_6^2(\triangle_{mn}^{(2)})$. $\square$

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