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NOTES ON NUNOKAWA LEMMAS

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Abstract: For analytic functions $p(z)$ in the open unit disk $\mathbb{U}$ with $p(0) = 1$, Nunokawa has given two results which are called Nunokawa lemmas (Proc. Japan Acad. Ser. A Math. Sci., **68** (1992), 152-153; Proc. Japan Acad. Ser. A Math. Sci., **69** (1993), 234-237). But, since then, nobody has given any examples related to these lemmas. The object of the present paper is to consider some simple and interesting examples for Nunokawa lemmas.

AMS Subject Classification: 30C45
Key Words: analytic function, Nunokawa lemma, Carathéodory function

1. Introduction

Let $\mathcal{N}$ denote the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For functions $p(z) \in \mathcal{N}$, Nunokawa [3, 4] has shown the following lemmas.

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Lemma 1. *Let $p(z) \in \mathcal{N}$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that* $\mathrm{Re}(p(z)) > 0$ $(|z| < |z_0|)$, $\mathrm{Re}(p(z_0)) = 0$ *and* $p(z_0) \neq 0$. *Then, we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where k is real and $|k| \geqq 1$.

Lemma 2. *Let $p(z) \in \mathcal{N}$ with $p(z) \neq 0$ in $\mathbb{U}$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg(p(z))| < \frac{\pi\alpha}{2} \quad \textit{for} \quad |z| < |z_0|$$

and

$$|\arg(p(z_0))| = \frac{\pi\alpha}{2},$$

where $\alpha > 0$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where

$$k \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \textit{when} \quad \arg(p(z_0)) = \frac{\pi\alpha}{2},$$

and

$$k \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \textit{when} \quad \arg(p(z_0)) = -\frac{\pi\alpha}{2},$$

where

$$p(z_0)^{\frac{1}{\alpha}} = \pm ia \qquad \textit{with} \quad a > 0.$$

The above two lemmas have been called *Nunokawa lemmas* and applied to obtain a number of interesting results by many mathematicians (see, for example, [1], [5]). But, nobody by now has provided some particular functions satisfying these lemmas. In this article, we obtain simple and interesting examples of Lemma 1 and Lemma 2, respectively.

2. Examples of Lemma 1

First, we consider an example for Lemma 1.

Example 1. Let us consider the function $p(z)$ defined by

$$p(z) = 1 + \frac{z}{1+iz}.$$

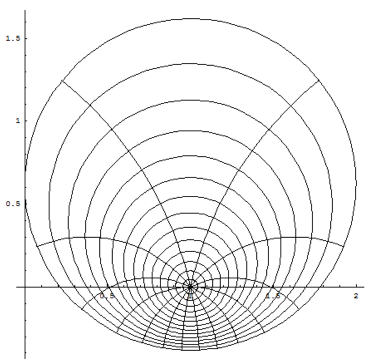
Then, it follows that $p(z) \in \mathcal{N}$, $\mathrm{Re}(p(z)) > 0$ $(|z| < |z_0|)$, $\mathrm{Re}(p(z_0)) = 0$ and $p(z_0) \neq 0$ for a point $z_0 = \dfrac{-2(1-2i)}{5+\sqrt{5}} \in \mathbb{U}$ $\left(|z_0| = \dfrac{\sqrt{5}-1}{2} < 1\right)$. Furthermore, we know that

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{5+\sqrt{5}}{2} i \equiv ik$$

and

$$k = \frac{5+\sqrt{5}}{2} = 3.618033\cdots \geqq 1.$$

Thus, $p(z)$ is the function satisfying Lemma 1. Indeed, $p(z)$ maps the circular domain $\{z : |z| < |z_0|\}$ onto the following domain.



We next discuss an example of Lemma 1 for the case that $p(z)$ maps the circular domain $\{z : |z| \leqq |z_0|\}$ onto the domain which touches the imaginary axis with two points.

Example 2. Let the function $p(z)$ be given by

$$p(z) = 1 + m\left(z + \frac{1}{2}z^2\right) \qquad \left(\frac{4}{3} < m < \frac{8}{3}\right).$$

For $z = re^{i\theta}$ $(0 < r < 1,\ \theta \in \mathbb{R})$, we have that

$$\begin{aligned}\mathrm{Re}(p(re^{i\theta})) &= 1 + mr\cos\theta + \frac{1}{2}mr^2\cos 2\theta \\ &= mr^2\cos^2\theta + mr\cos\theta + 1 - \frac{1}{2}mr^2.\end{aligned}$$

Setting $F(t) \equiv mr^2t^2 + mrt + 1 - \frac{1}{2}mr^2 \quad (-1 \leqq t = \cos\theta \leqq 1)$ and m is positive, we know that

$$F'(t_0) = mr(2rt_0 + 1) = 0 \quad \text{for} \quad t_0 = -\frac{1}{2r} < 0.$$

(i) For $0 < r \leqq \frac{1}{2}$ (*i.e.* $t_0 \leqq -1$), since $F'(t) \geqq 0$ in $[-1, 1]$,

$$F(t) \geqq F(-1) = \frac{1}{2}mr^2 - mr + 1 = 0$$

for $r = \dfrac{m - \sqrt{m(m-2)}}{m} \leqq \dfrac{1}{2}$. It follows from $m(m-2) \geqq 0$ and $r \leqq \frac{1}{2}$ that $m \geqq \frac{8}{3}$. Then, we obtain that $p(z_0) = 0$ for $z_0 = -\dfrac{m - \sqrt{m(m-2)}}{m}$. This is unsuitable for the example of the lemma.

(ii) For $\frac{1}{2} < r < 1$ (*i.e.* $-1 < t_0 < 0$), we derive that

$$F(t) \geqq F(t_0) = -\frac{1}{2}mr^2 + 1 - \frac{1}{4}m$$

and $F(t_0) = 0$ for $r = \sqrt{\dfrac{4-m}{2m}}$ $\left(t_0 = -\sqrt{\dfrac{m}{2(4-m)}}\right)$. Noting that $\frac{1}{2} < r = \sqrt{\dfrac{4-m}{2m}} < 1$, we see that $\frac{4}{3} < m < \frac{8}{3}$. Therefore, it follows that $\mathrm{Re}(p(z_0)) = 0$ $(p(z_0) \neq 0)$ and

$$\mathrm{Re}(p(z)) > 0 \qquad \left(|z| < |z_0| = \sqrt{\frac{4-m}{2m}}\right)$$

for $z_0 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\dfrac{8-3m}{m}}i$. Furthermore, simple computations give us that

$$p(z_0) = \pm\frac{\sqrt{m(8-3m)}}{4}i$$

and

$$z_0p'(z_0) = mz_0(1+z_0) = -\frac{4-m}{2},$$

that is, that

$$\frac{z_0p'(z_0)}{p(z_0)} = \pm\frac{2(4-m)}{\sqrt{m(8-3m)}}i \equiv ik^{\pm} \qquad \left(|k^{\pm}| = \frac{2(4-m)}{\sqrt{m(8-3m)}} \geqq 1\right).$$

This means that $p(z)$ is an example of Lemma 1. Indeed, taking $m = 2$, we have $p(z) = 1 + 2z + z^2$ which satisfies

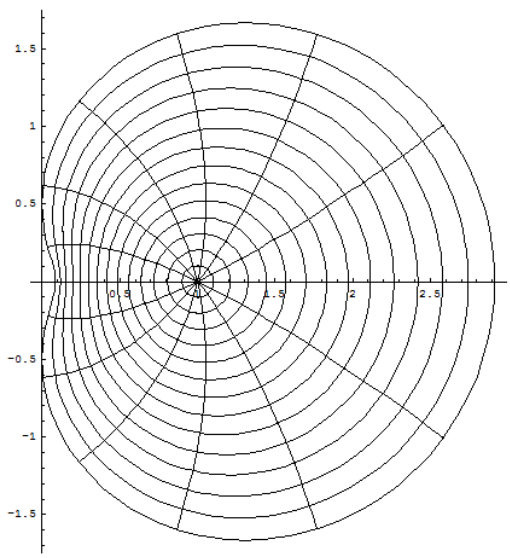
$$p(z_0) = \pm\frac{1}{2}i \neq 0 \qquad (\mathrm{Re}(p(z_0)) = 0),$$

$$\mathrm{Re}(p(z)) > 0 \qquad \left(|z| < |z_0| = \frac{1}{\sqrt{2}}\right)$$

and

$$\frac{z_0p'(z_0)}{p(z_0)} = \pm 2i \qquad (|k| = 2 \geqq 1)$$

for $z_0 = -\dfrac{1}{2} \pm \dfrac{1}{2}i$.



3. Examples of Lemma 2

In this section, we consider a function $p(z)$ satisfying Lemma 2 for every α $(0 < \alpha < 1)$.

Example 3. A function

$$p(z) = \frac{1+z}{1-z} \tag{3.1}$$

is an example of Lemma 2 for every α $(0 < \alpha < 1)$. Since $p(z)$ satisfies

$$\left| p(z) - \frac{1+r^2}{1-r^2} \right| = \frac{2r}{1-r^2} \qquad (|z| = r < 1)$$

which shows that $p(z)$ maps the circle $\{z : |z| = r\}$ onto the circle of center $\frac{1+r^2}{1-r^2}$ and radius $\frac{2r}{1-r^2}$, we know that

$$\mathrm{Re}(p(z)) > 0 \qquad (z \in \mathbb{U})$$

as $r \to 1^-$ and therefore, $p(z) \neq 0$ in $\mathbb{U}$. Let θ be the angle between the real axis and the tangent line of the above circle passing through the origin, and let $p(z_0)$ be the point of contact. Then, we establish

$$\theta = \pm \sin^{-1}\left(\frac{2r}{1+r^2}\right) \qquad \left(|\theta| = \sin^{-1}\left(\frac{2r}{1+r^2}\right) \equiv \frac{\pi\alpha}{2}\right)$$

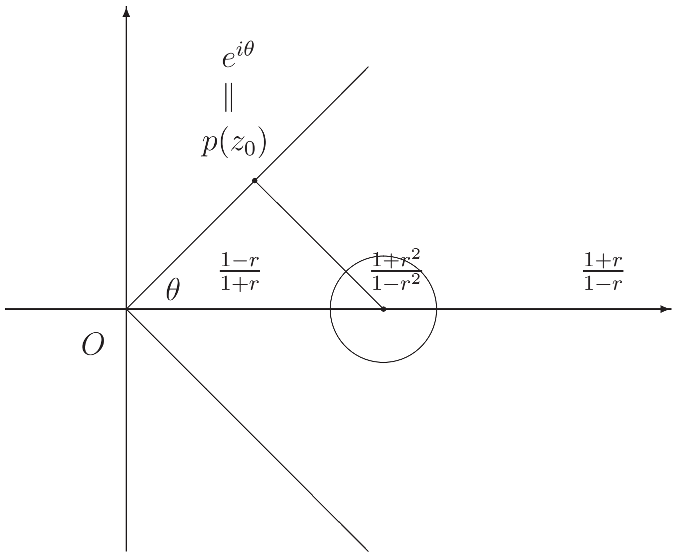
and

$$|p(z_0)| = \sqrt{\left(\frac{1+r^2}{1-r^2}\right)^2 - \left(\frac{2r}{1-r^2}\right)^2} = 1$$

for all r $(0 < r < 1)$. Namely, $p(z_0)$ can be written by

$$p(z_0) = e^{i\theta} \qquad \left(|\theta| < \frac{\pi}{2}\right).$$

Thus, every point $p(z_0)$ is on the right-side of the unit circle.



Since

$$p(z_0) = \frac{1+z_0}{1-z_0} = e^{i\theta} \qquad \left(\theta = \frac{\pi\alpha}{2} \text{ or } \theta = -\frac{\pi\alpha}{2}\right)$$

for some α $(0 < \alpha < 1)$, we obtain that

$$z_0 = \frac{-1+e^{i\theta}}{1+e^{i\theta}} = \frac{1-\cos\theta}{\sin\theta} i \quad \text{and} \quad |z_0| = \frac{1-\cos\theta}{|\sin\theta|}.$$

Furthermore, we also derive that

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\sin\theta \equiv ik\alpha \qquad \left(k = \frac{\pi\sin\theta}{2|\theta|}\right)$$

and

$$p(z_0)^{\frac{1}{\alpha}} = e^{i\frac{\pi}{2}\cdot\frac{\theta}{|\theta|}} = \pm i \equiv \pm ia \qquad (a = 1).$$

Then, it follows that

$$k = \frac{\pi\sin\theta}{2|\theta|} \geqq 1 = \frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \left(0 < \theta < \frac{\pi}{2}\right)$$

$$k = \frac{\pi\sin\theta}{2|\theta|} \leqq -1 = -\frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \left(-\frac{\pi}{2} < \theta < 0\right).$$

Therefore, $p(z)$ satisfies Lemma 2. Putting $\alpha = \dfrac{1}{3}$, we see that

$$\theta = \pm\frac{\pi}{6} = \arg(p(z_0^{\pm})), \quad p(z_0^{\pm}) = \frac{\sqrt{3} \pm i}{2}, \quad z_0^{\pm} = \pm(2-\sqrt{3})i$$

and

$$\frac{z_0^{\pm} p'(z_0^{\pm})}{p(z_0^{\pm})} = i\left(\pm\frac{3}{2}\right)\frac{1}{3} \equiv ik^{\pm}\alpha \qquad \text{(double sign corresponds)}$$

$$\left(k^{+} = \frac{3}{2}\ \left(\arg(p(z_0^{+})) = \frac{\pi}{6}\right), \quad k^{-} = -\frac{3}{2}\ \left(\arg(p(z_0^{-})) = -\frac{\pi}{6}\right)\right).$$

Finally, we note that

$$p(z_0^{\pm})^{\frac{1}{\alpha}} = \pm i \equiv \pm ia \qquad (a = 1),$$

$$k^{+} = \frac{3}{2} \geqq 1 = \frac{1}{2}\left(a + \frac{1}{a}\right)$$

and

$$k^{-} = -\frac{3}{2} \leqq -1 = -\frac{1}{2}\left(a + \frac{1}{a}\right).$$

4. Appendix

For some real parameters A and B $(-1 \leqq B < A \leqq 1)$, we introduce the following function

$$p(z) = \frac{1 + Az}{1 + Bz} \tag{4.1}$$

which is analytic and univalent in $\mathbb{U}$. This function $p(z)$ has been studied by Janowski [2] as the generalization function of (3.1) and therefore, it is said to be the Janowski function. The Janowski function $p(z)$ given by (4.1) satisfies the following equation

$$\left| p(z) - \frac{1 - ABr^2}{1 - B^2 r^2} \right| = \frac{(A - B)r}{1 - B^2 r^2} \qquad (|z| = r < 1)$$

which implies that $p(z)$ maps the circle $\{z : |z| = r\}$ onto the circle of center $\dfrac{1-ABr^2}{1-B^2r^2}$ and radius $\dfrac{(A-B)r}{1-B^2r^2}$ and

$$\mathrm{Re}(p(z)) > \frac{1-A}{1-B} \geqq 0 \qquad (z \in \mathbb{U}).$$

Thus, we discuss the same matters of Example 3 in this section. We first consider the case that $A \neq 0$ and $B \neq 0$.

Let θ be the angle between the real axis and the tangent line of the circle passing through the origin, and let $p(z_0)$ be the point of contact. Then, we see that

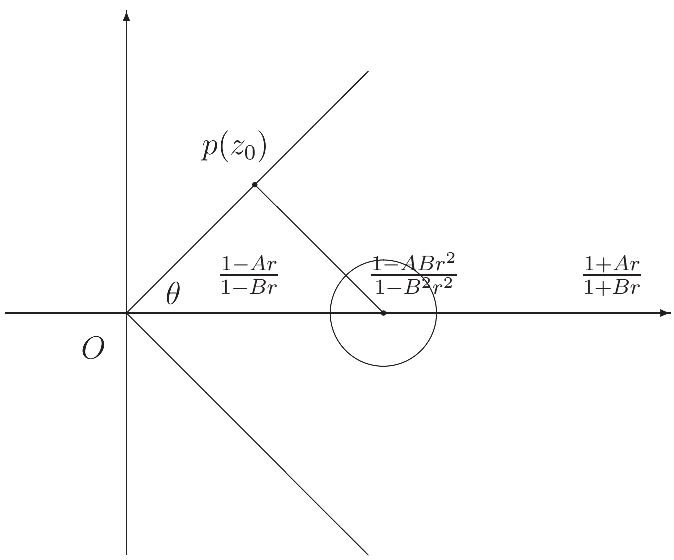
$$\theta = \pm \sin^{-1}\left(\frac{(A-B)r}{1-ABr^2}\right) \qquad \left(0 < |\theta| \equiv \frac{\pi\alpha}{2} < \frac{\pi}{2}\right)$$

which leads us that

$$r = \frac{-(A-B) + \sqrt{(A-B)^2 + 4AB\sin^2\theta}}{2AB\,|\sin\theta|}$$

where r is positive whether or not AB is positive. Furthermore, it follows that

$$p(z_0) = \frac{1+Az_0}{1+Bz_0} = \sqrt{\frac{1-A^2r^2}{1-B^2r^2}}e^{i\theta} \equiv Ce^{i\theta} \qquad \left(z_0 = \frac{-1+Ce^{i\theta}}{A-BCe^{i\theta}}\right).$$



We next need the description of C without r, so that

$$C^2 = \frac{A^2}{B^2} \cdot \frac{-2B\sin^2\theta - (A-B) + \sqrt{(A-B)^2 + 4AB\sin^2\theta}}{2A\sin^2\theta - (A-B) + \sqrt{(A-B)^2 + 4AB\sin^2\theta}}$$

$$= \frac{A^2}{B^2} \cdot \frac{-2B\sin^2\theta - (A-B) + \sqrt{(A-B)^2 + 4AB\sin^2\theta}}{2A\sin^2\theta - (A-B) + \sqrt{(A-B)^2 + 4AB\sin^2\theta}}$$

$$\cdot \frac{2A\sin^2\theta - (A-B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta}}{2A\sin^2\theta - (A-B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta}}$$

$$= \left(\frac{(A+B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta}}{2B\cos\theta} \right)^2$$

which equivalent to

$$C = \frac{\left|(A+B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta}\right|}{2|B|\cos\theta}.$$

If $A + B \leqq 0$, then

$$\left|(A+B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta}\right| = -(A+B) + \sqrt{(A-B)^2 + 4AB\sin^2\theta}.$$

Conversely, because if $A + B > 0$, then

$$(A+B)^2 - \left\{(A-B)^2 + 4AB\sin^2\theta\right\} = 4AB\cos^2\theta,$$

we can deduce that

$$(A+B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta} > 0 \qquad (AB > 0)$$

and

$$-(A+B) + \sqrt{(A-B)^2 + 4AB\sin^2\theta} > 0 \qquad (AB < 0).$$

Although we have to consider three cases (i) $0 < B < A$, (ii) $B < 0 < A$, (iii) $B < A < 0$, by virtue of the above facts, we obtain that

$$C = \frac{(A+B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta}}{2B\cos\theta}$$

in any case. We also derive that

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{(A-B)z_0}{(1+Az_0)(1+Bz_0)} = \frac{(-e^{-i\theta} + C)(A - BCe^{i\theta})}{(A-B)C}$$

and put $D \equiv (-e^{-i\theta}+C)(A-BCe^{i\theta})$. Then, we have that

$$\begin{aligned}
\operatorname{Re}(D) &= -A\cos\theta + (A+B)C - BC^2\cos\theta \\
&= -A\cos\theta + \frac{(A+B)^2-(A+B)\sqrt{(A-B)^2+4AB\sin^2\theta}}{2B\cos\theta} \\
&\quad -\frac{(A+B)^2+(A-B)^2+4AB\sin^2\theta-2(A+B)\sqrt{(A-B)^2+4AB\sin^2\theta}}{4B\cos\theta} = 0
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Im}(D) &= (A-BC^2)\sin\theta \\
&= \frac{4AB\cos^2\theta-(A+B)^2+(A+B)\sqrt{(A-B)^2+4AB\sin^2\theta}}{2B\cos^2\theta}\sin\theta.
\end{aligned}$$

Since $(A-B)C = \dfrac{(A-B)\left\{(A+B)-\sqrt{(A-B)^2+4AB\sin^2\theta}\right\}}{2B\cos\theta}$,

$$\begin{aligned}
&\frac{z_0p'(z_0)}{p(z_0)} \\
&= i\left(\frac{4AB\cos^2\theta-(A+B)^2+(A+B)\sqrt{(A-B)^2+4AB\sin^2\theta}}{(A-B)\left\{(A+B)-\sqrt{(A-B)^2+4AB\sin^2\theta}\right\}}\tan\theta\right) \\
&\equiv ik\alpha
\end{aligned}$$

$$\left(k=\frac{\pi}{2|\theta|}\cdot\frac{4AB\cos^2\theta-(A+B)^2+(A+B)\sqrt{(A-B)^2+4AB\sin^2\theta}}{(A-B)\left\{(A+B)-\sqrt{(A-B)^2+4AB\sin^2\theta}\right\}}\tan\theta\right).$$

Finally, we know that

$$p(z_0)^{\frac{1}{\alpha}} = \pm iC^{\frac{1}{\alpha}} \equiv \pm ia,$$

and consequently

$$a = C^{\frac{1}{\alpha}} = \left(\frac{(A+B)-\sqrt{(A-B)^2+4AB\sin^2\theta}}{2B\cos\theta}\right)^{\frac{\pi}{2|\theta|}} > 0.$$

Now, it is clear that $p(z)$ satisfies the conditions of Lemma 2. Thus, we expect that

$$k \geqq \frac{1}{2}\left(a+\frac{1}{a}\right) \qquad \left(0<\theta<\frac{\pi}{2}\right)$$

and

$$k \leqq -\frac{1}{2}\left(a+\frac{1}{a}\right) \qquad \left(-\frac{\pi}{2}<\theta<0\right).$$

But it is difficult to check it by the manual calculations for the general case.

In the same manner, we derive that

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\tan\theta \equiv ik\alpha \qquad \left(k=\frac{\pi}{2|\theta|}\tan\theta\right)$$

and

$$p(z_0)^{\frac{1}{\alpha}} = \pm i(\cos\theta)^{\frac{\pi}{2|\theta|}} \equiv \pm ia \qquad \left(a=(\cos\theta)^{\frac{\pi}{2|\theta|}}\right)$$

for the case $B=0$ $(0<A\leqq 1)$, and

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\tan\theta \equiv ik\alpha \qquad \left(k=\frac{\pi}{2|\theta|}\tan\theta\right)$$

$$p(z_0)^{\frac{1}{\alpha}} = \pm i(\cos\theta)^{-\frac{\pi}{2|\theta|}} \equiv \pm ia \qquad \left(a=(\cos\theta)^{-\frac{\pi}{2|\theta|}}\right)$$

for the case $A=0$ $(-1\leqq B<0)$. For these case, we can prove that

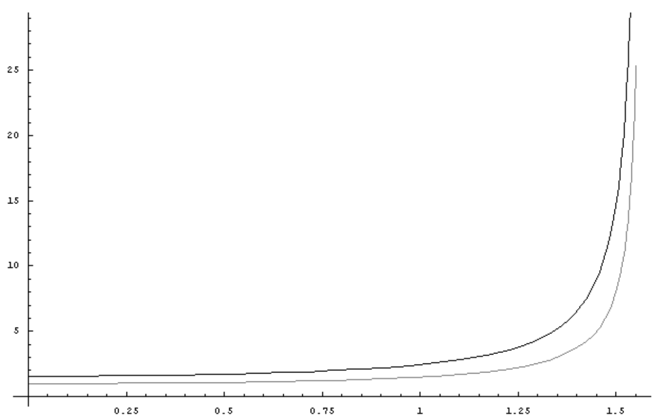
$$k \geqq \frac{1}{2}\left(a+\frac{1}{a}\right) \qquad \left(0<\theta<\frac{\pi}{2}\right)$$

and

$$k \leqq -\frac{1}{2}\left(a+\frac{1}{a}\right) \qquad \left(-\frac{\pi}{2}<\theta<0\right),$$

by using *Mathematica*.

For the particular case $B=-A$ $(0<A\leqq 1)$, we readily arrive at the same result of Example 3.



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