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LOCAL ESTIMATES FOR $L_n^{\alpha,\beta,M,N}(x;-1)$, $L_n^{\alpha,\beta,M,N}(x;1)$, POLYNOMIALS

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Abstract: In this note we give some local estimates for the $L_n^{\alpha,\beta,M,N}(x;-1)$, $L_n^{\alpha,\beta,M,N}(x;1)$, polynomials.

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1. Introduction

Let $\omega^{(\alpha,\beta)}(x) = (1-x)^\alpha \cdot (1+x)^\beta, x \in [-1,1]$, be a Jacobi weight with $\alpha, \beta > -1$. Let $p_n(x) = p_n^{(\alpha,\beta)}(x) = \gamma_n^{(\alpha,\beta)} x^n + \ldots, n \in \mathbb{N}_0$ denote the unique Jacobi polynomials of precise degree n, with leading coefficients $\gamma_n^{(\alpha,\beta)} > 0$, fulfilling the orthogonality conditions

$$\int_{-1}^{1} p_n(x) p_m(x) \omega^{(\alpha,\beta)}(x) = \delta_{m,n}, \quad n, m \in \mathbb{N}_0.$$

In [2], M. Felten, introduced modified Jacobi weights as

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$$\omega_n^{(\alpha,\beta)}(x) := \left(\sqrt{1-x}+\frac{1}{n}\right)^{2\alpha}\left(\sqrt{1+x}+\frac{1}{n}\right)^{2\beta}, \tag{1}$$

$x \in [-1,1], n \in \mathbb{N}$. He proved the following theorem (see [2]).

Theorem 1.1. *Let $\alpha, \beta > -1$ and $n \in \mathbb{N}$. Then*

$$|p_n^{(\alpha,\beta)}(x)| \le C\frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)} \tag{2}$$

for all $x \in [-1,1]$ with a positive constant $C = C(\alpha,\beta)$ being independent of n and x.

The above estimation first appeared in [1].

Then for $\alpha, \beta \ge -\frac{1}{2}$, Felten (see [2]), extended the previous result as follows:

Theorem 1.2. *Let $\alpha, \beta \ge -\dfrac{1}{2}$ and $n \in \mathbb{N}$. Then*

$$|p_n^{(\alpha,\beta)}(t)| \le C\frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)} \tag{3}$$

for all $t \in U_n(x)$ and each $x \in [-1,1]$, where

$$\begin{aligned} U_n(x) &:= \left\{t \in [-1,1] \mid |t-x| \le \frac{\varphi_n(x)}{n}\right\} \\ &= \left[x-\frac{\varphi_n(x)}{n}, x+\frac{\varphi_n(x)}{n}\right] \cap [-1,1] \end{aligned} \tag{4}$$

for $n \in \mathbb{N}$ and $x \in [-1,1]$ with $\varphi_n(x) := \sqrt{1-x^2}+\frac{1}{n}$.

In [5], T. H. Koornwinder, introduced the polynomials $P_n^{(\alpha,\beta,M,N)}(x)_{n=0}^{\infty}$ as follows:

Definition 1.3. Fix $M, N \ge 0$ and $\alpha, \beta > -1$. For $n = 0, 1, 2, \cdots$ define

$$\begin{aligned} &P_n^{(\alpha,\beta,M,N)}(x) \\ &= \left(\frac{(\alpha+\beta+1)_n}{n!}\right)^2 \cdot \left[(\alpha+\beta+1)^{-1}(B_nM(1-x) - A_nN(1+x)\frac{d}{dx} + A_nB_n)\right] \\ &\qquad p_n^{(\alpha,\beta)}(x), \end{aligned} \tag{5}$$

where

$$A_n = \frac{(\alpha+1)_n n!}{(\beta+1)_n(\alpha+\beta+1)_n} + \frac{n(n+\alpha\beta+1)M}{(\beta+1)(\alpha+\beta+1)}, \tag{6}$$

$$B_n = \frac{(\beta+1)_n n!}{(\alpha+1)_n(\alpha+\beta+1)_n} + \frac{n(n+\alpha\beta+1)N}{(\alpha+1)(\alpha+\beta+1)}. \tag{7}$$

We call these polynomials as Koornwinder's Jacobi-type polynomials.

The above defined polynomials are orthogonal on the interval $[-1,1]$ with respect to the measure μ defined by

$$\int_{-1}^{1} f(x)d\mu(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \cdot \int_{-1}^{1} f(x)(1-x)^{\alpha}(1+x)^{\beta}dx$$

$$+Mf(-1)+Nf(1), \tag{8}$$

where $f \in C([-1,1])$ and $M,N \geq 0, \alpha,\beta > -1$.

Clearly for $M = N = 0$ one has

$$P_n^{(\alpha,\beta,0,0)}(x) = P_n^{(\alpha,\beta)}(x). \tag{9}$$

Also

$$P_n^{(\alpha,\beta,M,N)}(-x) = (-1)^n P_n^{(\beta,\alpha,N,M)}(x). \tag{10}$$

Some basic properties of $P_n^{(\alpha,\beta,M,N)}(x)$ are given below (see [8], Chapter IV):

$$P_n^{(\alpha,\beta,M,N)}(1) \sim \begin{cases} n^{-\alpha-\frac{3}{2}}, & \text{if } N > 0 \\ n^{\alpha+\frac{1}{2}}, & \text{if } N = 0 \end{cases}, \tag{11}$$

$$|P_n^{(\alpha,\beta,M,N)}(-1)| \sim \begin{cases} n^{-\beta-\frac{3}{2}}, & \text{if } M > 0 \\ n^{\beta+\frac{1}{2}}, & \text{if } M = 0 \end{cases}. \tag{12}$$

Theorem 1.4. (see [8]) *Let $\alpha,\beta > -1, M,N > 0$. For every $x \in [-1,1]$ there exists a unique constant C such that the following relation*

$$\left(h_n^{\alpha,\beta,M,M}\right)^{-\frac{1}{2}} \left|P_n^{(\alpha,\beta,M,N)}(x)\right| \leq C\left(1-x+\frac{1}{n^2}\right)^{-\frac{\alpha}{2}-\frac{1}{4}} \cdot \left(1+x+\frac{1}{n^2}\right)^{-\frac{\beta}{2}-\frac{1}{4}}$$

holds for every $n \in \mathbb{N}$.

Based on Theorem 1.4 and the properties of Jacobian polynomials (see [1], [7]), we obtain the following estimation for the Koorwinder Jacobi-type polynomials:

$$|L_n^{(\alpha,\beta,M,N)}(x,-1)| \le C \begin{cases} n^{\alpha+\frac{1}{2}}, & \text{if } 0 \le \arccos x \le \frac{1}{n} \\ (\arccos x)^{-(\alpha+\frac{1}{2})}, & \text{if } \frac{1}{n} \le \arccos x \le \pi \end{cases} \tag{13}$$

for $\alpha \ge -1, \beta \ge -1$ and $n \ge 1$.

The basic tool in the estimation of the third term of (1) is Pollard's decomposition of the kernel $L_n^{\alpha,\beta,M,N}(x,y)$ in the form (see [6],[4])

$$L_n^{\alpha,\beta,M,N}(x,y) = r_n T_1(n,x,y) + s_n T_2(n,x,y) - s_n T_3(n,x,y),$$

where (r_n) and (s_n) are bounded sequences and

$$T_1(n,x,y) = p_{n+1}^{\alpha,\beta,M,N}(x) p_{n+1}^{\alpha,\beta,M,N}(y),$$

$$T_2(n,x,y) = (1-x^2)\frac{p_{n+1}^{\alpha,\beta,M,N}(x) p_{n+1}^{\alpha,\beta,M,N}(y)}{x-y},$$

$$T_3(n,x,y) = (1-y^2)\frac{p_{n+1}^{\alpha,\beta,M,N}(x) p_{n+1}^{\alpha,\beta,M,N}(y)}{x-y}.$$

The aim of this paper is to provide similar results as those in Theorem 1.1 and Theorem 1.2, for the $L_n^{\alpha,\beta,M,N}(x;-1)$,$L_n^{\alpha,\beta,M,N}(x;1)$ polynomials , when $\alpha,\beta \ge -1$, respectively for $\alpha,\beta > -\frac{1}{2}$.

2. Results

The following theorem is the main result of this note.

Theorem 2.1. *Let $\alpha,\beta > -1$, $n \in \mathbb{N}$ and $M > 0$. Then*

$$|L_n^{(\alpha,\beta,M,N)}(x,-1)| \le D \frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)} \tag{14}$$

for all $x \in [-1,1]$ with a positive constant $D = D(\alpha,\beta)$ being independent of n and x.

Proof. It is similar to that of Theorem 2.1 in [2]. Let $x \in [-1,1]$, and $\theta \in [0,\pi]$ such that $x = \cos\theta$. From (13) one has the following estimation

$$|L_n^{(\alpha,\beta,M,N)}(\cos\theta,-1)| \leq C \begin{cases} n^{\alpha+\frac{1}{2}}, & \text{if } 0 \leq \theta \leq \frac{1}{n} \\ \theta^{-(\alpha+\frac{1}{2})}, & \text{if } \frac{1}{n} \leq \theta \leq \pi \end{cases}. \tag{15}$$

If in the last relation, we substitute $x = \cos\theta$ we have:

$$|L_n^{(\alpha,\beta,M,N)}(x,-1)| \leq C \begin{cases} n^{\alpha+\frac{1}{2}}, 0 \leq \arccos x \leq \frac{1}{n} \\ (\arccos x)^{-(\alpha+\frac{1}{2})}, \frac{1}{n} \leq \arccos x \leq \pi \end{cases}, \tag{16}$$

where C is fixed positive constant being independent of n and θ. In what follows, we will make use of the following estimates:

$$\frac{\pi}{2}\sqrt{1-x} = \frac{\pi}{\sqrt{2}}\sqrt{\frac{1-x}{2}} = \frac{\pi}{\sqrt{2}}\sin\frac{t}{2} \geq \frac{\pi}{\sqrt{2}}\left(\frac{t}{16}\cdot\frac{(8-t)\pi}{\sqrt{2}}\right) \geq$$
$$t = \arccos x, \tag{17}$$

and

$$\sqrt{2}\sqrt{1-x} = 2\sqrt{\frac{1-x}{2}} = 2\sin\frac{t}{2} \leq 2\cdot\frac{t}{2} = t = \arccos x. \tag{18}$$

We differ two cases:

Case 1. $-1 < \alpha \leq -\frac{1}{2}$. In this case $-\left(\alpha+\frac{1}{2}\right) \geq 0$.

If $0 \leq \arccos x \leq \dfrac{c}{n}$, then from (18) we obtain $\dfrac{1}{n} \geq \sqrt{2}\sqrt{1-x}$ and from (15) we obtain

$$|L_n^{(\alpha,\beta,M,N)}(x,-1)| \leq Cn^{\alpha+\frac{1}{2}} = C\left(\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}$$
$$\leq C_1\left(\sqrt{1-x}\right)^{-\left(\alpha+\frac{1}{2}\right)} \leq C_2\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

If $\frac{1}{n} \leq \arccos x \leq \frac{\pi}{2}$, then from relations (16) and (17) we get

$$|L_n^{(\alpha,\beta,M,N)}(x,-1)| \leq C_3(\arccos x)^{-\left(\alpha+\frac{1}{2}\right)} \leq C_4\left(\sqrt{1-x}\right)^{-\left(\alpha+\frac{1}{2}\right)}$$
$$\leq C_5\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

Case 2. $\alpha > -\frac{1}{2}$: In this case $-\left(\alpha+\frac{1}{2}\right) < 0$.

If $0 \leq \arccos x \leq \frac{1}{n}$, then from relations (16) and (18) we obtain:

$$|L_n^{(\alpha,\beta,M,N)}(x,-1)| \leq C_6 n^{\alpha+\frac{1}{2}} = C_6 \left(\frac{c}{n} + \frac{\sqrt{2}}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}$$

$$\leq C_7 \left(\sqrt{1-x} + \frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

If $\frac{1}{n} \leq \arccos x \leq \frac{\pi}{2}$, again according to relations (16) and (18) we have:

$$|L_n^{(\alpha,\beta,M,N)}(x,-1)| \leq C_8 (\arccos x)^{-\left(\alpha+\frac{1}{2}\right)} = C_9 (\arccos x + \arccos x)^{-\left(\alpha+\frac{1}{2}\right)} \leq$$

$$C_{10} \left(\sqrt{1-x} + \frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

From the previous cases we have proved that:

$$|L_n^{(\alpha,\beta,M,N)}(x,-1)| \leq C_{11}(\alpha,\beta) \left(\sqrt{1-x} + \frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)} \cdot \left(\sqrt{1+x} + \frac{1}{n}\right)^{-\left(\beta+\frac{1}{2}\right)}$$

for all $x \in [-1,1], n \in \mathbb{N}$ and $\alpha, \beta \geq -1$. From (10) we obtain:

$$|L_n^{(\alpha,\beta,M,N)}(x,-1)| \leq C_{12}(\beta,\alpha) \left(\sqrt{1+x} + \frac{1}{n}\right)^{-\left(\beta+\frac{1}{2}\right)} \cdot \left(\sqrt{1-x} + \frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}$$

for all $x \in [-1,1), n \in \mathbb{N}$ and $\alpha, \beta \geq -1$. The proof is completed. $\square$

Next, we will show that the local estimates from the previous theorem can be further extended. We will prove that $|L_n^{(\alpha,\beta,M,N)}(x,-1)|$ in (14) can be replaced by $|L_n^{(\alpha,\beta,M,N)}(t,-1)|$, whenever t is in the interval $U_n(x) = \Big[x - \frac{\varphi_n(x)}{n}, x + \frac{\varphi_n(x)}{n}\Big] \cap [-1,1]$. In order to do that, we will make use of the following lemma (see [2]).

Lemma 2.2. *Let $a, b \leq 0, n \in \mathbb{N}$ and $x \in [-1,1]$. Then*

$$\omega_n^{(a,b)}(t) \leq 16^{-(a+b)} \omega_n^{(a,b)}(x) \tag{19}$$

for all $t \in U_n(x)$.

Theorem 2.3. *Let $\alpha, \beta \geq -\dfrac{1}{2}$ and $n \in \mathbb{N}$. Then*

$$|L_n^{(\alpha,\beta,M,N)}(t,-1)| \leq D \frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)} \tag{20}$$

for all $t \in U_n(x)$ and each $x \in [-1,1]$ where $D = D(\alpha,\beta)$ is a positive constant independent of n, t and x.

Proof. Since $\alpha, \beta \geq -\frac{1}{2}$ it follows that $\frac{\alpha}{2}+\frac{1}{4}, \frac{\beta}{2}+\frac{1}{4} \geq 0$. Therefore, by Lemma 2.2 with $a = -\frac{\alpha}{2}-\frac{1}{4}$ and $\beta = -\frac{\alpha}{2}-\frac{1}{4}$ we obtain

$$\frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)} = \omega_n^{\left(-\frac{\alpha}{2}-\frac{1}{4},-\frac{\beta}{2}-\frac{1}{4}\right)}(x) \leq \frac{4^{\alpha+\beta+1}}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)}$$

for all $t \in U_n(x)$. Applying Theorem 2.1 yields inequality (15) for all $t \in U_n(x)$, as claimed. □

Corollary 2.4. *For all $n \in \mathbb{N}$ with $\alpha, \beta \geq -\frac{1}{2}, x \in [-1,1]$ holds:*

$$\int_{U_n(x)} |L_n^{(\alpha,\beta,M,N)}(t,-1)|^2 \omega_n^{(\alpha,\beta)}(t)dt \leq D(\alpha,\beta) \cdot \frac{1}{n}. \tag{21}$$

Proof. Applying Theorem 2.3, we obtain:

$$\int_{U_n(x)} |L_n^{(\alpha,\beta,M,N)}(t,-1)|^2 \omega_n^{(\alpha,\beta)}(t)dt \leq D \cdot \frac{1}{\omega_n^{\left(\alpha+\frac{1}{2},\beta+\frac{1}{2}\right)}(x)} \cdot$$

$$\int_{U_n(x)} \omega_n^{(\alpha,\beta)}(t)dt.$$

Using the following result from [3] we obtain

$$\int_{U_n(x)} \omega_n^{(\alpha,\beta)}(t)dt \leq D \cdot \frac{1}{n} \cdot \omega_n^{\left(\alpha+\frac{1}{2},\beta+\frac{1}{2}\right)}(x)$$

and thus, the proof is completed. □

In a similar way, we can proof the result for $L_n^{\alpha,\beta,M,N}(x,1)$.

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