International Journal of Applied Mathematics

Volume 25 No. 3 2012, 451-456

RINGS IN WHICH POWER VALUES OF K-ENGELS WITH DERIVATIONS ANNIHILATE A CERTAIN ELEMENT

Shervin Sahebi[1 §], Venus Rahmani[2]

Department Of Mathematics
Islamic Azad University
Central Tehran Branch
13185.768, Tehran, IRAN
e-mail: sahebi@iauctb.ac.ir
e-mail: ven.rahmani.math@iauctb.ac.ir

Abstract: Let R be a 2-torsion free semiprime ring and d a non-zero derivation. Further let $A = O(R)$ be the orthogonal completion of R and $B = B(C)$ the Boolean ring of C where C be the extended centroid of R. We show that if $a[[d(x), x]_n, [y, d(y)]_m]^t = 0$ such that $0 \neq a \in R$ for all $x, y \in R$, where $m, n, t > 0$ are fixed integers, then there exists an idempotent $e \in B$ such that eA is a commutative ring and d induce a zero derivation on $(1 - e)A$.

AMS Subject Classification: 16R50, 16N60, 16D60
Key Words: prime ring, semiprime ring, derivation

1. Introduction

Let R be an associative ring with center $Z(R)$. Recall that an additive mapping d of R into itself is a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. Also if $(x_i)_{i\in\mathbb{N}}$ is a squence of elements of R and k is a positive integer, we define $[x_1, \ldots, x_{k+1}]$ inductively as follows:

$$[x_1, x_2] = x_1x_2 - x_2x_1, \quad [x_1, \ldots, x_k, x_{k+1}] = [[x_1, \ldots, x_k], x_{k+1}].$$

If $x_1 = x$ and $x_2 = \ldots = x_{k+1} = y$, the notation $[x, y]_k$ is used to denote $[x_1, \ldots, x_{k+1}]$ and $[x, y]_k$ is called a k-engel element.

Received: May 17, 2012

§Correspondence author

A well known result of Posner stated that if $[[d(x),x],y]=0$ for all $x,y\in R$, then R is commutative [13]. A number of authors extended this result in several ways. Bell and Martindale in [2] studied this identity for a semiprime ring R. They proved that if R is a semiprime ring and $[[d(x),x],y]=0$ for all x in a non-zero left ideal of R and $y\in R$, then R contains a non-zero central ideal. In [7], Filippis showed that if R is a prime ring with $charR\neq 2$ and d a non-zero derivation of R such that $[[d(x),x],[d(y),y]]=0$ for all $x,y\in R$, then R is commutative. Recently Dhara obtained results for a prime ring R of $charR\neq 2$, with a nonzero derivation d that if $0\neq a\in R$ such that $a[[d(x),x]_n,[d(y),y]_m]=0$ for all $x,y\in R$, where $m,n\geq 0$ are fixed integers, then R is commutative [5].

Now, we will generalize Posner's result [13] when the condition are more widespread.

The main result of this paper is as follows:

Theorem 1.1. *Let R be a 2-torsion free semiprime ring with non-zero derivation d and $0\neq a\in R$ such that $a[[d(x),x]_n,[y,d(y)]_m]^t=0$ for all $x,y\in R$, where $m,n,t>0$ are fixed integers. Further let $A=O(R)$ be the orthogonal completion of R and $B=B(C)$ where C the extended centroid of R. Then there exists an idempotent $e\in B$ such that eA is a commutative ring and d induce a zero derivation on $(1-e)A$.*

Throughout the paper we use the standard notation from [1]. In particular, we denote by Q the two sided Martindale quotient of prime and semiprime ring R and C the center of Q. We call C the extended centroid of R.

It is well known that any derivation of prime (semiprime) ring R can be uniquely extended to a derivation of Q, and so any derivation of R can be defined on the whole of Q. Moreover Q is a prime (semiprime) ring as well as R. We refer to [1, 11] for more details.

2. Proof of Main Result

The following results are usefull tool needed the proof of main result.

Theorem 2.1. *Let R be a prime ring of $charR\neq 2$ and d a derivation of R. Suppose $a[[d(x),x]_n,[d(y),y]_m]^t=0$ and $0\neq a\in R$ for all $x,y\in R$, where $m,n,t>0$ are fixed integers. Then R is commutative or $d=0$.*

Proof. Consider two cases.

Case 1. d is not a Q-inner derivation. By Kharchenko's Theorem [8] for any $x, y, z, s \in R$ we have $a[[z, x]_n, [s, y]_m]^t = 0$. This is a polynomial identity and hence there exists a field F such that $R \subseteq M_k(F)$ with $k > 1$ and $R, M_k(F)$ satisfy the same polynomial identity [10]. Therefore we can consider $a = (a_{ij})_{k\times k}$. We may assume that t is an even integer. Now putting

$$z = e_{ij}, \quad x = e_{ii}, \quad s = e_{ji}, \quad y = e_{ii}.$$

Thus for any $i \neq j$, we have

$$0 = a[[z, x]_n, [s, y]_m]^t = a(-1)^{nt}(e_{ii} + (-1)^t e_{jj}) = a(e_{ii} + e_{jj}),$$

This implies $a_{ij} = 0$ for any i, j $(i \neq j)$, which is contradiction.

Case 2. d is a Q-inner derivation. So there exists an element $b \in Q$ such that $d(x) = [b, x]$ for all $x \in R$. Since by [4] Q and R satisfy the same generalized polynomial identities (GPI), hence for any $x, y \in Q$ we have

$$a[[b, x]_{n+1}, [y, [b, y]]_m]^t = 0.$$

Also since Q remains prime by the primeness of R, replacing R by Q we may assume that $b \in R$ and the extended centroid of R is just the center of R. Note that R is a centrally closed prime C-algebra in the present situation [6]. If R is commutative, we have nothing to prove. So, let R be noncommutative. Therefore R satisfies a nontrivial (GPI). Since R is a centrally closed prime C-algebra, by Martindale's theorem [12], R is a strongly primitive ring. Let $_RV$ be a faitful irreducible left R-module with commuting ring $D = End(_RV)$. By the Density Theorem, R acts densely on V_D. For any given $v \in V$ we claim that v and bv are D-dependent. Assume first that $av \neq 0$. Suppose on the contrary that v and bv are D-independent.

If $b^2v \in span\{v, bv\}$, then $b^2v = v\alpha + bv\beta$ for some $\alpha, \beta \in D$. By density of R in $End(V_D)$ there exist two elements x and y in R such that

$$xv = v, xbv = 0 \quad \text{and} \quad yv = 0, ybv = v.$$

Then

$$0 = a[[b, x]_{n+1}, [y, [b, y]]_m]^t v = (-2)^{mt} av.$$

If $b^2v \notin span\{v, bv\}$, then $\{v, bv, b^2v\}$ are all D-independent. Then by Density of R in $End(V_D)$ there exist two elements x and y in R such that

$$xv = v, xbv = 0, xb^2v = 0 \quad \text{and} \quad yv = 0, ybv = 0, yb^2v = 0.$$

Therefore we have

$$0 = a[[b,x]_{n+1},[y,[b,y]]_m]^t v = (-2)^{mt} av.$$

Since $char R \neq 2$ we get $av = 0$, a contradiction. Thus v and bv are D-dependent as claimed. Assume next that $av = 0$. Since $a \neq 0$, we have $aw \neq 0$ for some $w \in V$. Then $a(v+w) = aw \neq 0$. Applying the first situation we have

$$bw = w\alpha \quad \text{and} \quad b(v+w) = (v+w)\beta,$$

for some $\alpha, \beta \in D$. But v and w are clearly D-independent, and so there exist two elements x and y in R such that

$$xw = w, xv = 0 \quad \text{and} \quad yw = v, yv = 0.$$

Then

$$0 = a[[b,x]_{n+1},[y,[b,y]]_m]^t = (-1)^{t(n+1)} 2^{mt} a(\beta - \alpha)^{2t} w,$$

which implies $\alpha = \beta$ and hence $bv = v\alpha$ as claimed. From the above we have proved that $bv = v\alpha(v)$ for all $v \in V$, where $\alpha(v) \in D$ depends on $v \in V$. In fact, it is easy to check that $\alpha(v)$ is independent of the choice of $v \in V$. That is, there exist $\delta \in D$ such that $bv = v\delta$ for all $v \in V$. we claim $\delta \in Z(D)$, the center of D. Indeed, if $\beta \in D$, then

$$b(v\beta) = (v\beta)\delta = v(\beta\delta)$$

and the other hand

$$b(v\beta) = (bv)\beta = (v\delta)\beta = v(\delta\beta).$$

Therefore $v(\beta\delta - \delta\beta) = 0$ so $\beta\delta = \delta\beta$, which implies $\delta \in Z(D)$. Thus $b \in C$ and hence $d = 0$, as be wanted. $\square$

The following example shows the hypothesis of primeness is essential in theorem 2.1.

Example 2.2. Let S be any ring, and

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a,b,c \in S \right\}.$$

Define $d : R \to R$ as follows:

$$d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $0 \neq d$ is a derivation of R such that $a[[d(x), x]_n, [d(y), y]_m]^t = 0$ for all $x, y \in R$, where $m, n, t > 0$ are fixed integers, however R is not commutative.

Now let R be a semiprime orthogonally complete ring with extended centeroid C. We use the notation $B = B(C)$ and spec(B) to denote Boolian ring of C and the set of all maximal ideal of B. It is well known that if $M \in spec(B)$ then $R_M = R/RM$ is prime [1, Theorem 3.2.7]. We refer to [1, pages 37, 38, 43, 120] for definations of Ω-Δ-ring, a first order formula of signature Ω-Δ, Horn formulas and Hereditary first order formulas.

In preparation for the proof of Theorem 1.1 we have the following lemma.

Lemma 2.3. (see [1, Theorem 3.2.18]) *Let R be an orthogonally complete Ω-Δ-ring with extended centroid C, $\Psi_i(x_1, x_2, ..., x_n)$ Horn formulas of signature Ω-Δ, $i = 1, 2, ...$ and $\Phi(y_1, y_2, ..., y_m)$ a Hereditary first order formula such that $\neg\Phi$ is a Horn formula. Further, let $\vec{a} = (a_1, a_2, ..., a_n) \in R^{(n)}$, $\vec{c} = (c_1, c_2, ..., c_m) \in R^{(m)}$. Suppose that $R \models \Phi(\vec{c})$ and for every $M \in spect(B)$ there exists a natural number $i = i(M) > 0$ such that*

$$R_M \models \Phi(\phi_M(\vec{c})) \Longrightarrow \Psi_i(\phi_M(\vec{a})),$$

where $\Phi_M : R \to R_M = R/RM$ is the canonical projection. Then there exist a natural number $k > 0$ and pairwise orthogonal idempotents $e_1, e_2, ..., e_k \in B$ such that $e_1 + e_2 + ... + e_k = 1$ and $e_i R \models \Psi_i(e_i \vec{a})$ for all $e_i \neq 0$.

Denote by $O(R)$ the orthogonal completion of R which is defined as the intersection of all orthogonally complete subset of Q containing R.

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. It is well known that the derivation d can be extended uniquely to a derivation $d : Q \to Q$. According to [1, Theorem 3.1.16] $d(A) \subseteq A$ and $d(e) = 0$ for all $e \in B$. Therefore A is an orthogonally complete Ω-Δ-ring where $\Omega = \{o, +, -, \cdot, d\}$. Consider formulas

$$\Phi = (\exists a \neq 0)(\forall x)(\forall y)\|a[[d(x), x]_n, [y, d(y)]_m]^t = 0\|,$$
$$\Psi_1 = (\forall x)(\forall y)\|xy = yx\|,$$
$$\Psi_2 = (\forall x)\|d(x) = 0\|.$$

One can easily check that Φ is a hereditary first order formula and $\neg\Phi$, Ψ_1, Ψ_2 are Horn formulas. So using Theorem 2.1 shows that all conditions of lemma 2.3 are fulfilled. Hence there exist two orthogonal idempotent e_1 and e_2 such that $e_1 + e_2 = 1$ and if $e_i \neq 0$, then $e_i A \models \Psi_i$, $i = 1, 2$. The proof is complete. $\square$

References

[1] K.I. Beidar, W.S. Martindale, A.V. Mikhalev, *Rings with Generalized Identities*, Pure and Applied Math., New York, Deloker.

[2] H.E. Bell, W.S. Martindale III, Centeralizing mappings of semiprime rings, *Canadian Mathematical Bulletin*, **30**, No. 1 (1987), 92-101.

[3] L. Carini, V.D. Filippis, Commutators with power central values on lie ideals, *Pacific J. Math.*, **193**, No. 2 (2000), 296-278.

[4] C.L. Chung, GPIs having coefficients in Utumi quotient rings, *Proc. Amer. Math. Soc.*, **103** (1988), 723-728.

[5] B. Dhara, On the annihilators of derivations with Engel conditions in prime rings, *Tamsui Oxford Jurnal of Mathematical Scienes*, **26**, No. 3 (2010), 255-264.

[6] J.S. Ericson, W.S. Martindale 3rd, J.M. Osborn, Prime nonassociative algebras, *Pascific J. Math.*, **60** (1975), 49-63.

[7] V.De. Filippis, On derivations and commutativity in prime rings, *Int. J. Math. Sci.*, **70** (2004), 3859-3856.

[8] V.K. Kharchenko, Differential identity of prime rings, *Algebra and Logic.*, **17** (1978), 155-168.

[9] C. Lanski, Differential identities Lie ideals and Posner's theorems, *Pacific Journal of Mathematics*, **134**, No. 2 (1988), 275-297.

[10] C. Lanski, An Engel condition with derivation, *Proc. Amer. Math. Soc.*, **118** (1993), 731-734.

[11] T.K. Lee, Semiprime rings with differential identities, *Bull. Inst. Math. Acad. Sinica*, **20**, No. 1 (1992), 27-38.

[12] W.S. Martindale 3rd, Prime rings satistying a generalized polynomial identity, *J. Algebra*, **12** (1969), 576-584.

[13] E.C. Posner, Derivation in prime rings, *Proc. Amer. Math. Soc.*, **8** (1957), 1093-1100.