International Journal of Applied Mathematics

Volume 25 No. 4 2012, 457-471

CONTRA $g\delta s$-CONTINUOUS FUNCTIONS IN TOPOLOGICAL SPACES

S.S. Benchalli[1], Umadevi I. Neeli[2], Gurupad P. Siddapur[3 §]

[1,3]Department of Mathematics
Karnatak University
Dharwad, 580 003, Karnataka State, INDIA
[1]e-mail: benchalliss@gmail.com
[3]e-mail: siddapur.math@gmail.com

[2]Department of Mathematics
BVB College of Engineering and Technology
Hubli, 580031, Karnataka State, INDIA
e-mail: uineeli@bvb.edu

Abstract: In this paper, the notion of $g\delta s$-open sets in topological space is applied to present and study a new class of functions called contra $g\delta s$-continuous functions as a new generalization of contra continuity and obtain their characterizations and properties. Also, the relationships with some other related functions are discussed.

AMS Subject Classification: 54C08, 54C10

Key Words: $g\delta s$-closed, $g\delta s$-open, $g\delta s$-connected, contra $g\delta s$-continuous, $g\delta s$-continuous

1. Introduction

Many topologists studied various types of generalizations of continuity, see e.g. [1, 8, 12, 14, 15]. In 1996, Dontchev [9] introduced the notion of contra continuity and strong S-closedness in topological spaces. A new weaker form of

Received: February 10, 2012

§Correspondence author

this class of functions called contra semi continuous function is introduced and investigated by Dontchev and Noiri [10]. Caldas and Jafari [6] introduced and studied the contra β-continuous functions and contra almost β-continuity is introduced and investigated by Baker [2]. In this paper, the notion of $g\delta s$-open sets in topological space is applied to introduce and study a new class of functions called contra $g\delta s$-continuous functions, as a new generalization of contra continuity, and to obtain some of their characterizations and properties. Also, the relationships with some other functions are discussed.

2. Preliminaries

Throughout this paper (X, τ), (Y, σ) and (Z, η) (or simply X, Y and Z) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X the closure and interior of A with respect to τ are denoted by $Cl(A)$ and $Int(A)$ respectively. A subset A of X is called regular open (resp. regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). The δ-interior [20] of a subset A of X is the union of all regular open sets of X contained in A and is denoted by δ-$Int(A)$ and the set A of X is called δ-open [20] if A=δ-$Int(A)$. The complement of a δ-open set is called δ-closed.

Definition 2.1. A subset A of a space X is called:

(1) a semiopen set [13] if $A \subset Cl(Int(A))$.

(2) an α-open set [16] if $A \subset Int(Cl(Int(A)))$.

(3) a regular open set [19] if $A = Int(Cl(A))$.

The complements of the above mentioned sets are called their respective closed sets. The semi-closure [7] of a subset A of a space X is the intersection of all semiclosed sets that contain A and is denoted by $sCl(A)$. The semi interior [7] of a subset A of space X is the union of all semiopen sets contained in A and is denoted by $sInt(A)$.

Definition 2.2. ([3]) A subset A of X is $g\delta s$-closed if $sCl(A) \subset U$ whenever $A \subset U$ and U is δ-open in X. The family of all $g\delta s$-closed subsets of the space X is denoted by $G\delta SC(X)$.

Definition 2.3. ([3]) The intersection of all $g\delta s$-closed sets containing a set A is called $g\delta s$-closure of A and is denoted by $g\delta s$-$Cl(A)$.

A set A is $g\delta s$-closed set if and only if $g\delta s$-$Cl(A) = A$.

Definition 2.4. [3] The union of all $g\delta s$-open sets contained in A is called $g\delta s$-interior of A and is denoted by $g\delta s$-$Int(A)$.

A set A is $g\delta s$-open if and only if $g\delta s$-$Int(A) = A$.

Definition 2.5. ([3]) A topological space X is called:

(1) ${}_{\delta}T_{3/4}$[3] if every $g\delta s$-closed subset of X is δ-closed.

(2) $g\delta sT_{1/2}$[3] if every $g\delta s$-closed subset of X is semiclosed.

(3) $Tg\delta s$[4] if every $g\delta s$-closed subset of X is closed.

Definition 2.6. ([4]) A function $f : X \to Y$ is called $g\delta s$-continuous, if the inverse image of every closed set in Y is $g\delta s$-closed in X.

3. Contra $g\delta s$-Continuous Functions

In this section, the notion of a new class of functions called contra $g\delta s$-continuous functions is introduced and we obtain some of their characterizations and properties. Also, the relationships with some other related functions are discussed.

Definition 3.1. A function $f : X \to Y$ is said to be contra $g\delta s$-continuous (resp., contra continuous [9], contra semi continuous [10]) if $f^{-1}(V)$ is $g\delta s$-closed (resp., closed, semiclosed) in X for each open set V of Y.

Definition 3.2. [17] A function $f : X \to Y$ is said to be almost continuous if $f^{-1}(V)$ is open in X for each regular open set V of Y.

Remark 3.3. From the following examples, it is clear that both contra $g\delta s$-continuous and $g\delta s$-continuous are independent notions of each other.

Example 3.4. Let $X = Y=\{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}\}$ be topologies on X and Y respectively. Define a function $f : X \to Y$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$. Then f is $g\delta s$-continuous function but not contra $g\delta s$-continuous, because for the open set $\{a, b\}$ in Y, $f^{-1}(\{a, b\}) =\{a, b\}$ is not $g\delta s$-closed in X.

Example 3.5. Let $X = Y=\{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \ \{a, c\}\}$ be topologies on X and Y respectively. Define a function $f : X \to Y$ by $f(a) = c$, $f(b) = b$ and $f(c) = a$. Then f is contra $g\delta s$-continuous function but not $g\delta s$-continuous, because for the open set $\{a\}$ in Y, $f^{-1}(\{a\})$ $=\{c\}$ is not $g\delta s$-open in X.

Theorem 3.6. *If $f : X \to Y$ is contra continuous, then it is contra $g\delta s$-continuous.*

Proof. Let V be an open set in Y. Since f is contra continuous, $f^{-1}(V)$ is closed in X. Since every closed set is $g\delta s$-closed, $f^{-1}(V)$ is $g\delta s$-closed in X. Therefore f is contra $g\delta s$-continuous.

Remark 3.7. Converse of the above theorem need be true in general as seen from the following example.

Example 3.8. Let $X = Y=\{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$,and $\sigma = \{Y, \phi, \{a\}, \ \{a, c\}\}$ be topologies on X and Y respectively. Define a function $f : X \to Y$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then f is contra $g\delta s$-continuous function but not contra-continuous, because for the open set $\{a\}$ in Y and $f^{-1}(\{a\})$ $=\{b\}$ is not closed in X.

Theorem 3.9. *If $f : X \to Y$ is contra semi continuous, then it is contra $g\delta s$-continuous.*

Proof. Let V be an open set in Y. Since f is contra semi continuous, $f^{-1}(V)$ is semiclosed in X. Since every semiclosed set is $g\delta s$-closed, $f^{-1}(V)$ is $g\delta s$-closed in X. Therefore f is contra $g\delta s$-continuous.

Remark 3.10. Converse of the above theorem need be true in general as seen from the following example.

Example 3.11. Let $X = Y=\{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, c\}\}$, and $\sigma = \{Y, \phi, \{a\}, \ \{b\}, \{a, b\}\}$ be topologies on X and Y respectively. Define a function $f : X \to Y$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then f is contra $g\delta s$-continuous function but not contra semi continuous, because for the open set $\{b\}$ in Y and $f^{-1}(\{b\})$ $=\{a\}$ is not semiclosed in X.

Lemma 3.12. [11] *The following properties hold for subsets A and B of a space X:*

(i) $x \in ker(A)$ if and only if $A \cap F \neq \phi$ for any closed set F of X containing x.

(ii) $A \subset ker(A)$ and $A = ker(A)$ if A is open in X.

(iii) If $A \subset B$, then $ker(A) \subset ker(B)$.

Theorem 3.13. *The following are equivalent for a function $f : X \to Y$:*

(i) f is contra $g\delta s$-continuous.

(ii) for every closed set F of Y, $f^{-1}(F)$ is $g\delta s$-open set of X.

(iii) for each $x \in X$ and each closed set F of Y containing $f(x)$, there exists $g\delta s$-open set U containing x such that $f(U) \subset F$

(iv) for each $x \in X$ and each open set V of Y not containing $f(x)$, there exists $g\delta s$-closed set K not containing x such that $f^{-1}(V) \subset K$.

(v) $f(g\delta s\text{-}Cl(A)) \subset ker(f(A))$ for every subset A of X.

(vi) $g\delta s\text{-}Cl(f^{-1}(B)) \subset f^{-1}(ker(B))$ for every subset B of Y.

Proof. $(i) \Rightarrow (ii)$ Let F be a closed set in Y. Then $Y - F$ is an open set in Y. By (i), $f^{-1}(Y - F)$=$X - f^{-1}(F)$ is $g\delta s$-closed set in X. This implies $f^{-1}(F)$ is $g\delta s$-open set in X. Therefore (ii) holds.

$(ii) \Rightarrow (i)$ Let G be an open set of Y. Then $Y - G$ is a closed set in Y. By (ii), $f^{-1}(Y - G) = X - f^{-1}(G)$ is $g\delta s$-open set in X, which implies $f^{-1}(G)$ is $g\delta s$-closed set in X. Therefore (i) hold.

$(ii) \Rightarrow (iii)$ Let F be a closed set in Y containing $f(x)$. Then $x \in f^{-1}(F)$. By (ii), $f^{-1}(F)$ is $g\delta s$-open set in X containing x. Let $U = f^{-1}(F)$. Then $f(U) = f(f^{-1}(F)) \subset F$. Therefore (iii) holds.

$(iii) \Rightarrow (ii)$ Let F be a closed set in Y containing $f(x)$. Then $x \in f^{-1}(F)$. From (iii), there exists $g\delta s$-open set U_x in X containing x such that $f(U_x) \subset F$. That is $U_x \subset f^{-1}(F)$. Thus $f^{-1}(F)$ $=\cup \left\{U_x : x \in f^{-1}(F)\right\}$, which is union of $g\delta s$-open sets. Since union of $g\delta s$-open sets is a $g\delta s$-open set, $f^{-1}(F)$ is $g\delta s$-open set of X.

$(iii) \Rightarrow (iv)$ Let V be an open set in Y not containing $f(x)$. Then $Y - V$ is closed set in Y containing $f(x)$. From (iii), there exists a $g\delta s$-open set U in X containing x such that $f(U) \subset Y - V$. This implies $U \subset f^{-1}(Y - V)$=$X -$

$f^{-1}(V)$. Hence, $f^{-1}(V) \subset X - U$. Set $K = X - U$, then K is $g\delta s$-closed set not containing x in X such that $f^{-1}(V) \subset K$.

$(iv) \Rightarrow (iii)$ Let F be a closed set in Y containing $f(x)$. Then $Y - F$ is an open set in Y not containing $f(x)$. From (iv), there exists $g\delta s$-closed set K in X not containing x such that $f^{-1}(Y - F) \subset K$. This implies $X - f^{-1}(F) \subset K$. Hence, $X - K \subset f^{-1}(F)$, that is $f(X - K) \subset F$. Set $U = X - K$, then U is $g\delta s$-open set containing x in X such that $f(U) \subset F$.

$(ii) \Rightarrow (v)$ Let A be any subset of X. Suppose $y \notin ker(f(A))$. Then by lemma 3.12, there exists a closed set F in Y containing y such that $f(A) \cap F = \phi$. Thus, $A \subset f^{-1}(F) = \phi$. Therefore $A \subset X - f^{-1}(F)$. By (ii), $f^{-1}(F)$ is $g\delta s$-open set in X and hence $X - f^{-1}(F)$ is $g\delta s$-closed set in X. Therefore, $g\delta s$-$Cl(X - f^{-1}(F)) = X - f^{-1}(F)$. Now $A \subset X - f^{-1}(F)$, which implies $g\delta s$-$Cl(A) \subset g\delta s$-$Cl(X - f^{-1}(F)) = X - f^{-1}(F)$. Therefore $g\delta s$-$Cl(A) \cap f^{-1}(F) = \phi$, which implies $f(g\delta s$-$Cl(A)) \cap F = \phi$ and hence $y \notin g\delta s$-$Cl(A)$. Therefore $f(g\delta s$-$Cl(A)) \subset ker(f(A))$ for every subset A of X.

$(v) \Rightarrow (vi)$ Let $B \subset Y$. Then $f^{-1}(B) \subset X$. By (iv) and Lemma 3.12, $f(g\delta s$-$Cl(f^{-1}(B))) \subset ker(f(f^{-1}(B))) \subset Ker(B)$. Thus $g\delta s$-$Cl(f^{-1}(B)) \subset f^{-1}(ker(B))$ for every subset B of Y.

$(vi) \Rightarrow (i)$ Let V be any open subset of Y. Then by (vi) and Lemma 3.12, $g\delta s$-$Cl(f^{-1}(V) \subset f^{-1}(ker(V)) = f^{-1}(V)$ and $g\delta s$-$Cl(f^{-1}(V)) = f^{-1}(V)$. Therefore $f^{-1}(V)$ is $g\delta s$-closed set in X. This shows that f is contra $g\delta s$-continuous.

Theorem 3.14. *If a function $f : X \to Y$ is contra $g\delta s$-continuous and Y is regular, then f is $g\delta s$-continuous.*

Proof. Let $x \in X$ and V be an open set in Y containing $f(x)$. Since Y is regular, there exists an open set W in Y containing $f(x)$ such that $Cl(W) \subset V$. Since f is contra $g\delta s$-continuous, by Theorem 3.13 (iii), there exists $g\delta s$-open set U in X containing x such that $f(U) \subset Cl(W)$. Then $f(U) \subset Cl(W) \subset V$. Therefore f is $g\delta s$-continuous.

Theorem 3.15. *If a function $f : X \to Y$ is contra $g\delta s$-continuous and X is $Tg\delta s$-space, then f is contra continuous.*

Proof. Let U be an open set in Y. Since f is contra $g\delta s$-continuous, $f^{-1}(U)$ is $g\delta s$-closed in X. Since X is T$g\delta s$-space, $f^{-1}(U)$ is a closed set in X. Therefore f is contra continuous.

Theorem 3.16. *If a function $f : X \to Y$ is contra $g\delta s$-continuous and X is $g\delta sT_{1/2}$ space, then f is contra semi continuous.*

Proof. Let U be an open set in Y. Since f is contra $g\delta s$-continuous, $f^{-1}(U)$ is $g\delta s$-closed in X. Since X is $g\delta sT_{1/2}$ space, $f^{-1}(U)$ is a semiclosed set in X. Therefore f is contra semi continuous.

Definition 3.17. A space X is called locally $g\delta s$-indiscrete if every $g\delta s$-open set is closed in X.

Theorem 3.18. *If a function $f : X \to Y$ is contra $g\delta s$-continuous and X is locally $g\delta s$-indiscrete space, then f is continuous.*

Proof. Let U be an open set in Y. Since f is contra $g\delta s$-continuous and X is locally $g\delta s$-indiscrete space, $f^{-1}(U)$ is an open set in X. Therefore f is continuous.

Definition 3.19. A function $f : X \to Y$ is called almost $g\delta s$-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in G\delta SO(X, x)$ such that $f(U) \subset Int(Cl(V))$.

Definition 3.20. A function $f : X \to Y$ is said to be quasi $g\delta s$-open if image of every $g\delta s$-open set of X is open set in Y.

Theorem 3.21. *If a function $f : X \to Y$ is contra $g\delta s$-continuous, quasi $g\delta s$-open, then f is almost $g\delta s$-continuous function.*

Proof. Let x be any arbitrary point of X and V be an open set in Y containing $f(x)$. Then $Cl(V)$ is a closed set in Y ontaining $f(x)$. Since f is contra $g\delta s$-continuous, then by Theorem 3.13 (iii), there exists $U \in G\delta SO(X, x)$ such that $f(U) \subset Cl(V)$. Since f is quasi $g\delta s$-open, $f(U)$ is open in Y. Therefore $f(U)=Int(f(U))$. Thus, $f(U) \subset Int(Cl(V))$. This shows that f is almost $g\delta s$-continuous function.

Definition 3.22. A function $f : X \to Y$ is called weakly $g\delta s$-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in G\delta SO(X, x)$ such that $f(U) \subset Cl(V)$.

Theorem 3.23. *If a function $f : X \to Y$ is contra $g\delta s$-continuous, then f is weakly $g\delta s$-continuous function.*

Proof. Let V be an open set in Y. Since $Cl(V)$ is closed in Y, by theorem 3.13 (ii), $f^{-1}(Cl(V))$ is $g\delta s$-open set in X. Set $U = f^{-1}(Cl(V))$, then $f(U) \subset f(f^{-1}(Cl(V))) \subset Cl(V)$. This shows that f is almost weakly $g\delta s$-continuous function.

Definition 3.24. Let A be a subset of X. Then $g\delta s$-$Cl(A) - g\delta s$-$Int(A)$ is called $g\delta s$-frontier of A and is denoted by $g\delta s$-$Fr(A)$.

Theorem 3.25. *The set of all points of x of X at which $f : X \to Y$ is not contra $g\delta s$-continuous is identical with the union of $g\delta s$-frontier of the inverse images of closed sets of Y containing $f(x)$.*

Proof. Assume that f is not contra $g\delta s$-continuous at $x \in X$. Then by theorem 3.13 (iii), there exists $F \in C(Y, f(x))$ such that $f(U) \cap (Y - F) \neq \phi$ for every $U \in G\delta SO(X, x)$. This implies $U \cap f^{-1}(Y - F) \neq \phi$ for every $U \in G\delta SO(X, x)$. Therefore, $x \in g\delta s$-$Cl(f^{-1}(Y - F))$= $g\delta s$-$Cl(X - f^{-1}(F))$. Also $x \in f^{-1}(F) \subset g\delta s$-$Cl(f^{-1}(F))$. Thus, $x \in g\delta s$-$Cl(f^{-1}(F)) \cap g\delta s$-$Cl(X - f^{-1}(F))$. This implies $x \in g\delta s$-$Cl(f^{-1}(F))$-$g\delta s$-$Int(f^{-1}(F))$. Therefore, $x \in g\delta s$-$Fr(f^{-1}(F))$.

Conversely, suppose $x \in g\delta s$-$Fr(f^{-1}(F))$ for some $F \in C(Y, f(x))$ and f is contra $g\delta s$-continuous at $x \in X$, then there exists $U \in G\delta SO(X, x)$ such that $f(U) \subset F$. Therefore, $x \in U \subset f^{-1}(F)$ and hence $x \in g\delta s$-$Int(f^{-1}(F)) \subset X - g\delta s$-$Fr(f^{-1}(F))$. This contradicts the fact that $x \in g\delta s$-$Fr(f^{-1}(F))$. Therefore f is not contra $g\delta s$-continuous.

Theorem 3.26. *Let $f : X \to Y$ be a function and let $g : X \times X \to Y$ be the graph function of f defined by $g(x)$=$(x, f(x))$ for every $x \in X$. If g is contra $g\delta s$-continuous, then f is contra $g\delta s$-continuous.*

Proof. Let U be an open set in Y. Then $X \times U$ is an open set in $X \times Y$. Since g is contra $g\delta s$-continuous, it follows that $f^{-1}(U) = g^{-1}(X \times U)$ is a $g\delta s$-closed set in X. Therefore f is contra $g\delta s$-continuous.

Theorem 3.27. *Assume $G\delta SO(X)$ is closed under any intersection. If $f : X \to Y$and $g : X \to Y$ are contra $g\delta s$-continuous and Y is Urysohn, then*

$E = \{x \in X : f(x) = g(x)\}$ is $g\delta s$-closed in X.

Proof. Let $x \in X - E$. Then $f(x) \neq g(x)$. Since Y is Urysohn, there exist open sets V and W such that $f(x) \in V$, $g(x) \in W$ and $Cl(V) \cap Cl(W)=\phi$. Since f and g are contra $g\delta s$-continuous, $f^{-1}(Cl(V))$ and $g^{-1}(Cl(W))$ are $g\delta s$-open sets in X. Let $U = f^{-1}(Cl(V))$ and $G = g^{-1}(Cl(W))$. Then U and G are $g\delta s$-open sets containing x. Set $A = U \cap G$, thus A is $g\delta s$-open set in X. Hence $f(A) \cap g(A)=f(U \cap G) \cap g(U \cap G) \subset f(U) \cap g(G)= Cl(V) \cap Cl(W)=\phi$. Therefore, $A \cap E=\phi$. This implies $x \notin g\delta s\text{-}Cl(E)$. Hence E is $g\delta s$-closed set in X.

Definition 3.28. A subset A of a topological space X is said to be $g\delta s$-dense in X if $g\delta s\text{-}Cl(A)=X$.

Theorem 3.29. *Assume $G\delta SO(X)$ is closed under any intersection. If $f : X \to Y$ and $g : X \to Y$ are contra $g\delta s$-continuous, Y is Urysohn, and $f=g$ on $g\delta s$-dense set $A \subset X$, then $f = g$ on X.*

Proof. Since f and g are contra $g\delta s$-continuous, Y is Urysohn, by the previous Theorem 3.2.7, $E = \{x \in X : f(x) = g(x)\}$ is $g\delta s$-closed in X. By assumption, $f = g$ on $g\delta s$-dense set A subset of X. Since $A \subset E$ and A is $g\delta s$-dense set in X, then $X = g\delta s\text{-}Cl(A) \subset g\delta s\text{-}Cl(E)=E$. Hence $f = g$ on X.

Definition 3.30. A space X is called $g\delta s$-connected provided that X is not the union of two disjoint nonempty $g\delta s$-open sets.

Theorem 3.31. *If $f : X \to Y$ is a contra $g\delta s$-continuous from a $g\delta s$-connected space X onto any space Y, then Y is not a discrete space.*

Proof. Let $f : X \to Y$ is a contra $g\delta s$-continuous and X is $g\delta s$-connected space. Suppose Y is a discrete space. Let A be a proper non empty open and closed subset of Y. Then $f^{-1}(A)$ is a proper nonempty $g\delta s$-open and $g\delta s$-closed subset of X, which is contradiction to the fact that X is $g\delta s$-connected space. Therefore, Y is not a discrete space.

Theorem 3.32. *If $f : X \to Y$ is a contra $g\delta s$-continuous surjection and X is $g\delta s$-connected space, then Y is connected.*

Proof. Let $f : X \to Y$ is a contra $g\delta s$-continuous and X is $g\delta s$-connected space. Suppose Y is a not connected space. Then there exist disjoint open sets U and V such that $Y = U \cup V$. Therefore U and V are clopen in Y. Since f is contra $g\delta s$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $g\delta s$-open sets in X. Further f is surjective implies, $f^{-1}(U)$ and $f^{-1}(V)$ are non empty disjoint and $X = f^{-1}(U) \cup f^{-1}(V)$. This is contradiction to the fact that X is $g\delta s$-connected space. Therefore, Y is connected.

Theorem 3.33. *Let X be a $g\delta s$-connected and Y be T_1-space, If $f : X \to Y$ is contra $g\delta s$-continuous, then f is constant.*

Proof. Let $f : X \to Y$ is contra $g\delta s$-continuous, X be a $g\delta s$-connected and Y is T_1. Since Y is T_1-space, $\Delta = \left\{f^{-1}(y) : y \in Y\right\}$ is a disjoint $g\delta s$-open partition of X. If $|\Delta| \geq 2$, then there exists a proper $g\delta s$-open and $g\delta s$-closed set W. This is contradiction to the fact that X is $g\delta s$-connected. Therefore$|\Delta| = 1$ and hence f is constant.

Definition 3.34. ([5]) A topological space X is said to be $g\delta s$-T_2 space if for any pair of distinct points x and y, there exist disjoint $g\delta s$-open sets G and H such that $x \in G$ and $y \in H$.

Theorem 3.35. *Let X and Y be topological spaces. If*

(i) for each pair of distinct points x and y in X, there exists a function $f : X \to Y$ such that $f(x) \neq f(y)$,

(ii) Y is an Urysohn space,

(iii) f is contra $g\delta s$-continuous at x and y.

Then X is $g\delta s$-T_2.

Proof. Let x and y be any distinct points in X and $f : X \to Y$ is a function such that $f(x) \neq f(y)$. Let $a = f(x)$ and $b = f(y)$, then $a \neq b$. Since Y is an Urysohn space, there exist open sets V and W in Y containing a and b respectively, such that $Cl(V) \cap Cl(W)=\phi$. Since f is contra $g\delta s$-continuous at x and y, then there exist $g\delta s$-open sets A and B in X containing x and y, respectively, such that $f(A) \subset Cl(V)$ and $f(B) \subset Cl(W)$. Then $f(A) \cap f(B) \subset Cl(V) \cap Cl(W) = \phi$. Therefore $A \cap B = \phi$. Hence X is $g\delta s$-T_2.

Corollary 3.36. *Let $f : X \to Y$ be contra $g\delta s$-continuous injective function from space X into an Urysohn space Y, then X is $g\delta s$-T_2.*

Proof. For each pair of distinct points x and y in X, f is contra $g\delta s$-continuous function from a space X into a Urysohn space such that that $f(x) \neq f(y)$ because f is injective. Hence by Theorem 3.35, X is $g\delta s$-T_2.

Definition 3.37. ([18]) A topological space X is called Ultra Hausdorff space, if for every pair of distinct points x and y in X, there exist disjoint clopen sets U and V in X containing x and y respectively.

Theorem 3.38. *If $f : X \to Y$ be contra $g\delta s$-continuous injective function from space X into a Ultra Hausdorff space Y, then X is $g\delta s$-T_2.*

Proof. Let x and y be any two distinct points in X. Since f is injective $f(x) \neq f(y)$ and Y is Ultra Hausdorff space, implies there exist disjoint clopen sets U and $V$of Y containing $f(x)$ and $f(y)$ respectively. Then $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$, where $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint $g\delta s$-open sets in X. Therefore X is $g\delta s$-T_2.

Definition 3.39. ([18]) A topological space X is called Ultra normal space, if each pair of disjoint closed sets can be separated by disjoint clopen sets.

Definition 3.40. ([5]) A topological space X is said to be $g\delta s$-normal if each pair of disjoint closed sets can be separated by disjoint $g\delta s$-open sets.

Theorem 3.41. *If $f : X \to Y$ be contra $g\delta s$-continuous closed injection and Y is ultra normal, then X is $g\delta s$-normal.*

Proof. Let E and F be disjoint closed subsets of X. Since f is closed and injective $f(E)$ and $f(F)$ are disjoint closed sets in Y. Since Y is ultra normal there exists disjoint clopen sets U and V in Y such that $f(E) \subset U$ and $f(F) \subset V$. This implies $E \subset f^{-1}(U)$ and $F \subset f^{-1}(V)$. Since f is contra $g\delta s$-continuous injection, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint $g\delta s$-open sets in X. This shows X is $g\delta s$-normal.

Theorem 3.42. *If $f : X \to Y$ is contra $g\delta s$-continuous and $g : Y \to Z$ is* *continuous. Then $g \circ f : X \to Z$ is contra $g\delta s$-continuous.*

Proof.: Let V be any open set in Z. Since g is continuous $g^{-1}(V)$ is open in Y. Since f is contra $g\delta s$-continuous $f^{-1}(g^{-1}(V))$=$(g \circ f)^{-1}(V)$ is $g\delta s$-closed set in X. Therefore, $g \circ f$ is contra $g\delta s$-continuous.

Theorem 3.43. *Let $f : X \to Y$ be contra $g\delta s$-continuous and $g : Y \to Z$ be $g\delta s$-continuous. If Y is $Tg\delta s$-space, then $g \circ f : X \to Z$ is contra $g\delta s$-continuous.*

Proof. Let V be any open set in Z. Since g is $g\delta s$-continuous, $g^{-1}(V)$ is $g\delta s$-open in Y and since Y is $Tg\delta s$-space $g^{-1}(V)$ open in Y. Since f is contra $g\delta s$-continuous $f^{-1}(g^{-1}(V))$=$(g \circ f)^{-1}(V)$ is $g\delta s$-closed set in X. Therefore, $g \circ f$ is contra $g\delta s$-continuous.

Definition 3.44. ([5]) A function $f : X \to Y$ is said to be strongly $g\delta s$-open (resp. strongly $g\delta s$-closed) if image of every $g\delta s$-open (resp. $g\delta s$-closed) set of X is $g\delta s$-open (resp. $g\delta s$-closed) set in Y.

Theorem 3.45. *If $f : X \to Y$ is surjective strongly $g\delta s$-open (or strongly $g\delta s$-closed) and $g : Y \to Z$ is a function such that $g \circ f : X \to Z$ is contra $g\delta s$-continuous, then g is contra $g\delta s$-continuous.*

Proof. Let V be any closed (resp. open) set in Z. Since $g \circ f$ is contra $g\delta s$-continuous, $(g \circ f)^{-1}(V)$=$f^{-1}(g^{-1}(V)$ is $g\delta s$-open (resp. $g\delta s$-closed). Since f is surjective and strongly $g\delta s$-open (or strongly $g\delta s$-closed), $f(f^{-1}(g^{-1}(V))$=$g^{-1}(V)$ is $g\delta s$- open(or $g\delta s$-closed). Therefore g is contra $g\delta s$-continuous.

Definition 3.46. A space X is said to be:

(i) $G\delta S$-closed compact if every $g\delta s$-closed cover of X has a finite subcover.

(ii) Countably $G\delta S$-closed compact if every countable cover of X by $g\delta s$-closed sets has a finite subcover.

(iii) $G\delta S$-Lindeloff if every $g\delta s$-closed cover of X has a countable subcover.

Theorem 3.47. *Let $f : X \to Y$ be a contra $g\delta s$-continuous surjection. Then, the following properties hold:*

(i) If X is $G\delta S$-closed compact, then Y is compact.

(ii) If X is countably $G\delta S$-closed compact, then Y is countably compact.

(iii) If X is $G\delta S$-Lindeloff, then Y is Lindeloff.

Proof. (i) Let $\{V_\alpha : \alpha \in I\}$ be an open cover of Y. Since f is contra $g\delta s$-continuous, then $\left\{f^{-1}(V_\alpha) : \alpha \in I\right\}$ is $g\delta s$-closed cover of X. Since X is $G\delta S$-closed compact, there exists a finite subset I_0 of I such that $X = \cup\left\{f^{-1}(V_\alpha) : \alpha \in I_0\right\}$. Since f is surjective, $Y = \cup\{V_\alpha) : \alpha \in I_0\}$, which is finite subcover for Y. Therefore, Y is compact.

(ii) Let $\{V_\alpha : \alpha \in I\}$ be any countable open cover of Y. Since f is contra $g\delta s$-continuous, then $\left\{f^{-1}(V_\alpha) : \alpha \in I\right\}$ is countable $g\delta s$-closed cover of X. Since X is countably $G\delta S$-closed compact, there exists a finite subset I_0 of I such that $X = \cup\left\{f^{-1}(V_\alpha) : \alpha \in I_0\right\}$. Since f is surjective, $Y = \cup\{V_\alpha) : \alpha \in I_0\}$ is finite subcover for Y. Therefore, Y is countably compact.

(iii) Let $\{V_\alpha : \alpha \in I\}$ be an open cover of Y. Since f is contra $g\delta s$-continuous, then $\left\{f^{-1}(V_\alpha) : \alpha \in I\right\}$ is $g\delta s$-closed cover of X. Since X is $G\delta S$-Lindeloff, there exists a countable subset I_0 of I such that $X = \cup\left\{f^{-1}(V_\alpha) : \alpha \in I_0\right\}$. Since f is surjective, $Y = \cup\{V_\alpha) : \alpha \in I_0\}$ is finite subcover for Y. Therefore, Y is Lindeloff.

Acknowledgements

The authors are grateful to the University Grants Commission, New Delhi – India for its financial support under UGC SAP II to the Department of Mathematics, Karnatak University, Dharwad, India and also for UGC-Research Fellowship under RFSMS Scheme, to the third author.

References

[1] M.E. Abd El-Monsef, S.N. El-Deeb, R.A. Mahmud, β-open sets and β-continuous mappings, *Bull. Fac. Sci. Assint Univ.*, **12** (1983), 77-90.

[2] C.W. Baker, On contra almost β-continuous functions, *Kochi J. Math.*, **1**(2006), 1-8.

[3] S.S. Benchalli, Umadevi I. Neeli, Generalized δ-semi closed sets in topological spaces, *International Journal of Applied Mathematics*, **24**, No. 1 (2011), 21-38.

[4] S.S. Benchalli, Umadevi I. Neeli, Generalized δ-semi continuous functions in topological spaces, *Communicated.*

[5] S.S. Benchalli, Umadevi I. Neeli, On gδs-separation axioms, *Communicated.*

[6] M. Caldas, S. Jafari, Some properties of contra β-continuous functions, *Memories of the Faculty of Sciences Kochi University, Series A. Mathematics*, **22** (2001), 19-28.

[7] S.G. Crossley, S.K. Hildebrand, Semi-topological properties, *Fund. Math.*, **74** (1972), 233-254.

[8] J. Dontchev, M. Przemski, On various decomposition of continuous and some weekly continuous functions, *Math. Hungar*, **71** (1996), 109-120.

[9] J. Dontchev, Contra continuous functions and strongly S-closed mappings, *Int. J. Math. Sci.*, **19** (1996), 303-310.

[10] J. Dontchev, T. Noiri, Contra semi continuous functions, *Mathematica Pannonica.*, **10**, No. 2 (1999), 159-168.

[11] S. Jafari, T. Noiri, On contra precontinuous functions, *Bull. of Malaysian Mathematical Sci. Soc.*, **25** (2002), 115-128.

[12] N. Levine, A decomposition of continuity in topological spaces, *Mer. Math.Monthly*, **68** (1961), 44-66.

[13] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Mer. Math. Monthly*, **70** (1963), 36-41

[14] A.S. Mashhour, M.E. Abd El-Monsef, S.N. EL-Deeb, On pre-continuous and weak pre-continuous mappings, *Proc. Math and Phys. Soc. Egypt*, **53** (1982), 47-53.

[15] A.S. Mashhour, I.A. Hasanein, S.N. EL-Deeb, α-continuous and α-open mappings, *Acta Math. Hung*, **41**, No-s: 3-4 (1983), 213-218.

[16] O. Njastad, On some classes of nearly open sets, *Pacific. J. Math.*, **15** (1965), 961-970.

[17] T. Noiri, On almost continuous functions, *TInd. J. of Pure and Appl. Maths.*, **20** (1989), 571-576.

[18] R. Staum, The algebra of bounded continuous functions into a non-archimedean field, *Pacific J. Math.*, **50** (1974), 169-185.

[19] M. Stone, Applications of the theory of boolean rings to general topology, *Trans. Amer. Math. Soc.*, **41** (1937), 374-481.

[20] N.V. Velicko, H-closed topological spaces, *Amer. Math. Soc. Transl.*, **78** (1968), 103-118.