International Journal of Applied Mathematics

Volume 25 No. 4 2012, 571-582

USING BLOCK PULSE FUNCTIONS FOR SOLVING TWO-DIMENSIONAL FUZZY FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

Azim Rivaz[1 §], Farzaneh Yousefi[2], Hojjat Salehinejad[3]

[1,2]Department of Mathematics
Shahid Bahonar University of Kerman
Kerman, 76169-14111, IRAN
[1]e-mail: azimrivaz91@yahoo.com
[2]e-mail: fyousefi89@yahoo.com

[3]Deparment of Electrical Engineering
Shahid Bahonar University of Kerman
Kerman, IRAN
e-mail: H.salehi@eng.uk.ac.ir

Abstract: In this paper, a numerical method based on block pulse functions (BPFs) is presented for solving a linear two dimensional fuzzy Fredholm integral equation of the second kind (2D-FFIE-2). We use parametric form of fuzzy functions and convert a 2D-FFIE-2 to a linear system of Fredholm integral equations of the second kind with three variables in crisp case. In this work, block pulse functions are used to find the approximate solution of the converted system, which is the approximate solution for 2D-FFIE-2. To evaluate performance of the algorithm, some numerical examples are presented which implicate the accuracy of the method.

AMS Subject Classification: 45BXX, 45FXX

Key Words: two-dimensional fuzzy Fredholm integral equation, parametric form of fuzzzy integral equation, block pulse functions

Received: July 8, 2012

[§]Correspondence author

1. Introduction

The concept of integration of fuzzy functions was first introduced by Dubois and Prade. Alternative approaches were later suggested by Goetschel and Voxman [9], Kaleva [8], Matloka [6], Nanda [10] and others. One of the first applications of fuzzy integration was given by Wu and Ma (Wu et al. 1990), who investigated the fuzzy Fredholm integral equation of the second kind (FFIE-2).

In recent years, some numerical methods have been introduced to solve (FFIE-2). These methods can be found in Friedman et al. 1999; Babolian et al. 2005, Abbasbandy et al. 2006; Goghary et al. 2006; Ghanbari et al. 2009, and others. Also, some numerical methods are presented to solve two-dimensional fuzzy Fredholm integral equations of the second kind. As an example, in [1] Adomian decomposition method and homotopy perturbation method are applied to solve a 2D-FFIE-2.

The main purpose of this work is to use one-dimensional and two-dimensional block pulse functions for solving 2D-FFIE-2. Block pulse functions (BPFs) are studied by many authors and applied for solving different problems; for example, see [2]. The piece wise constant BPFs are easy to use and this simplicity allows one to use them to solve integral equations and differential equations. In [5] Ghanbari et al. apply BPFs for solving FFIE-2. In this work, we use BPFs for solving 2D-FFIE-2. The remainder of this paper is organized as follows: A review of the block pulse functions and the fuzzy numbers is presented in Sections 2 and 3, respectively. In Section 4, two-dimensional fuzzy Fredholm integral equation and its parametric form are discussed. Then, the BPFs expansion is presented in Section 5. Later, the proposed algorithm is discussed in detail in Section 6. The algorithm is numerically discussed in Section 7 and finaly the paper is concluded in Section 8.

2. Block Pulse Functions

2.1. One-Dimensional Block Pulse Functions (1D-BPFs)

An m-set of 1D-BPFs is defined over the interval $[0, 1)$ as [3]:

$$\phi_i(t) = \begin{cases} 1 & (i-1)h \leq t < ih, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

where $i = 1, \ldots, m$ with a positive integer value for m. Also, consider $h = 1/m$, and ϕ_i is the $i$the 1D-BPF.

The most important properties of 1D-BPFs are: disjointness, orthogonality and completeness.

1. Disjointness

$$\phi_i(t)\phi_j(t) = \begin{cases} \phi_i(t) & i = j, \\ 0 & i \neq j, \end{cases} \tag{2}$$

where $i, j = 1, \dots, m$.

2. Orthogonality

$$\int_0^1 \phi_i(t)\phi_j(t)dt = h\delta ij, \quad t \in [0, 1), \tag{3}$$

where δij is kroneker delta.

3. Completeness. For every $f \in L^2([0, 1))$, parseval's identity holds:

$$\int_0^1 f^2(t)dt = \sum_{i=1}^{\infty} f_i^2 \|\phi_i(t)\|^2, \tag{4}$$

where

$$f_i = m \int_0^1 f(t)\phi_i(t)dt. \tag{5}$$

Consider the first m terms of 1D-BPFs and write them consisely as a m-vector $\Phi(t)$:

$$\Phi(t) = [\phi_1(t), \dots, \phi_m(t)]^T, \quad t \in [0, 1). \tag{6}$$

2.2. Two-Dimensional Block Pulse Functions (2D-BPFs)

An (m_1m_2)-set of 2D-BPFs is defined in the region of $x \in [0, 1)$ and $y \in [0, 1)$ as [4]:

$$\phi_{i_1,i_2}(x, y) = \begin{cases} 1 & (i_1 - 1)h_1 \leq x < i_1h_1 \text{ and } (i_2 - 1)h_2 \leq y < i_2h_2, \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

where $i_1 = 1, \dots, m_1$ and $i_2 = 1, \dots, m_2$ with positive integer values for m_1, m_2 and $h_1 = \frac{1}{m_1}, h_2 = \frac{1}{m_2}$.

Similar to 1D cases, there are some properties for 2D-BPFs as follows:

1. Disjointness

$$\phi_{i_1,i_2}(x, y)\phi_{j_1,j_2}(x, y) = \begin{cases} \phi_{i_1,i_2}(x, y) & i_1 = j_1 \text{ and } i_2 = j_2, \\ 0 & \text{otherwise}. \end{cases} \tag{8}$$

2. Orthogonality

$$\int_0^1 \int_0^1 \phi_{i_1,i_2}(x,y)\phi_{j_1,j_2}(x,y)dxdy = \begin{cases} h_1h_2 & i_1 = j_1 \text{ and } i_2 = j_2, \\ 0 & \text{otherwise}, \end{cases} \tag{9}$$

where $i_1, j_1 = 1, \dots, m_1$ and $i_2, j_2 = 1, \dots, m_2$.

3. Completeness. For every $f \in L^2([0,1)\times[0,1))$ when m_1 and m_2 approach to the infinity, parseval's identity holds:

$$\int_0^1 \int_0^1 f^2(x,y)dxdy = \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty} f_{i_1,i_2}^2 \|\phi_{i_1,i_2}(x,y)\|^2, \tag{10}$$

where

$$f_{i_1,i_2} = m_1m_2 \int_0^1 \int_0^1 f(x,y)\phi_{i_1,i_2}(x,y)dxdy. \tag{11}$$

The set of 2D-BPFs may be written as a (m_1m_2)-vector $\Phi(x,y)$:

$$\Phi(x,y) = [\phi_{1,1}(x,y), \dots, \phi_{1,m_2}(x,y), \dots, \phi_{m_1,1}(x,y), \dots, \phi_{m_1,m_2}]^T. \tag{12}$$

From the above representation and disjiontness property, it follows:

$$\Phi(x,y)\Phi(x,y)^T = \begin{pmatrix} \phi_{1,1}(x,y) & 0 & \dots & 0 \\ 0 & \phi_{1,2}(x,y) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \phi_{m_1,m_2}(x,y) \end{pmatrix}, \tag{13}$$

also

$$\int_0^1 \int_0^1 \Phi(x,y)\Phi(x,y)^T dxdy = h_1h_2I, \tag{14}$$

where I is an (m_1m_2) identity matrix.

3. Fuzzy Numbers

Definition 1. The parametric form of a fuzzy number u is a pair of functions $(\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$, which satisfies the following requirements:
1. $\underline{u}(r)$ is a bounded, continuous, monotonic increasing function over $[0,1]$.
2. $\overline{u}(r)$ is a bounded, continuous, monotonic decreasing function over $[0,1]$.
3. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.
$(\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$, are called the r-cut sets of u.

The set of all fuzzy numbers is denoted by E^1.

Definition 2. For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$ and real number k, we have

$$(u+v)(r) = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)),$$

$$(u-v)(r) = (\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r)),$$

$$(ku)(r) = \begin{cases} (k\underline{u}(r), k\overline{u}(r)) & k \geq 0, \\ (k\overline{u}(r), k\underline{u}(r)) & k < 0. \end{cases}$$

Definition 3. For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ the quantity

$$D(u,v) = \max\left\{ \sup_{0 \leq r \leq 1} |\underline{u}(r) - \underline{v}(r)|, \sup_{0 \leq r \leq 1} |\overline{u}(r) - \overline{v}(r)| \right\},$$

is called the distance between u and v.

It is shown that (E^1, D) is a complete metric space, see [6].

Definition 4. A function $f : R^2 \to E^1$ is called a fuzzy function in two-dimensional space. f is said to be continuous, if for arbitrary fixed $t_0 \in R^2$ and $\varepsilon > 0$ a $\delta > 0$ exists such that

$$\|t - t_0\| < \delta \Rightarrow D(f(t), f(t_0)) < \varepsilon, \qquad t = (x,y),\ t_0 = (x_0, y_0).$$

Definition 5. Let $f : [a,b] \times [c,d] \to E^1$.
For each partition $p = \{x_1, x_2, \ldots, x_m\}$ of $[a,b]$ and $q = \{y_1, y_2, \ldots, y_n\}$ of $[c,d]$ and for arbitrary $\xi_i : x_{i-1} \leq \xi_i \leq x_i$, $2 \leq i \leq m$, and for arbitrary $\eta_j : y_{j-1} \leq \eta_j \leq y_j$, $2 \leq j \leq n$, let

$$R_p = \sum_{i=2}^{m} \sum_{j=2}^{n} f(\xi_i, \eta_j)(x_i - x_{i-1})(y_j - y_{j-1}).$$

The definite integral of $f(x,y)$ over $[a,b] \times [c,d]$ is

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \lim R_p,$$

$$(\max_{2\le i\le m}|x_i - x_{i-1}|, \max_{2\le j\le n}|y_j - y_{j-1}|) \to (0,0),$$

provided that this limit exists in metric D.

If the function $f(x,y)$ is continuous in the metric D, its definite integral exists [7]. Furthermore

$$\left(\underline{\int_c^d \int_a^b f(x,y,r)dxdy}\right) = \int_c^d \int_a^b \underline{f}(x,y,r)dxdy,$$

$$\left(\overline{\int_c^d \int_a^b f(x,y,r)dxdy}\right) = \int_c^d \int_a^b \overline{f}(x,y,r)dxdy.$$

4. Two-Dimensional Fuzzy Integral Equation

The linear two-dimensional fuzzy Fredholm integral equation of the second kind (2D-FFIE-2) is

$$u(x,y) = f(x,y) + \int_c^d \int_a^b k(x,y,s,t)u(s,t)dsdt, \quad (x,y) \in V, \tag{15}$$

where $u(x,y)$ and $f(x,y)$ are fuzzy functions on $V = [a,b]\times[c,d]$, and $k(x,y,s,t)$ is an arbitrary kernel function over $S = [a,b] \times [c,d] \times [a,b] \times [c,d]$, and u is unknown on V.

Now suppose that $(\underline{f}(x,y,r), \overline{f}(x,y,r))$ and $(\underline{u}(x,y,r), \overline{u}(x,y,r))$, $0 \le r \le 1$, $(x,y) \in V$ are parametric forms of fuzzy functions $f(x,y)$ and $u(x,y)$, respectively, then parametric form of 2D-FFIE-2 is as follows:

$$\begin{aligned} \underline{u}(x,y,r) &= \underline{f}(x,y,r) + \textstyle\int_c^d \int_a^b \upsilon_1(x,y,s,t,\underline{u}(s,t,r),\overline{u}(s,t,r))dsdt, \\ \overline{u}(x,y,r) &= \overline{f}(x,y,r) + \textstyle\int_c^d \int_a^b \upsilon_2(x,y,s,t,\underline{u}(s,t,r),\overline{u}(s,t,r))dsdt, \end{aligned} \tag{16}$$

$$\upsilon_1(x,y,s,t,\underline{u}(s,t,r),\overline{u}(s,t,r)) = \begin{cases} k(x,y,s,t)\underline{u}(s,t,r) \ k(x,y,s,t) \ge 0, \\ k(x,y,s,t)\overline{u}(s,t,r) \ k(x,y,s,t) < 0, \end{cases}$$

and

$$\upsilon_2(x,y,s,t,\underline{u}(s,t,r),\overline{u}(s,t,r)) = \begin{cases} k(x,y,s,t)\overline{u}(s,t,r) \ k(x,y,s,t) \ge 0, \\ k(x,y,s,t)\underline{u}(s,t,r) \ k(x,y,s,t) < 0. \end{cases}$$

for each $a \leq x \leq b$ and $c \leq y \leq d$ and $0 \leq r \leq 1$. We can see that (16) is a system of linear Fredholm integral equations of the second kind with three variables in crisp case. So we use the mentioned algoritm in Section 6 to solve it.

5. BPFs Expansion

The expansion of a function $f(t)$ over $[0,1)$ with respect to 1D-BPFs may be compactly written as

$$f(t) \simeq \sum_{i=1}^{m} f_i \phi_i(t) = F^T \Phi(t) = \Phi(t)^T F, \tag{17}$$

where $\Phi(t)$ is m-component 1D-BPFs vector and $F = [f_1, \ldots, f_m]^T$, where f_i's are defined by (5).

Also a function $f(x,y) \in L^2([0,1) \times [0,1))$ can be expanded by 2D-BPFs as

$$f(x,y) \simeq \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} f_{i_1,i_2} \phi_{i_1,i_2}(x,y) = F^T \Phi(x,y) = \Phi(x,y)^T F, \tag{18}$$

where $F = [f_{1,1}, \ldots, f_{1,m_2}, \ldots, f_{m_1,1}, \ldots, f_{m_1,m_2}]^T$, and f_{i_1,i_2}'s are defined by (11).

Similarly a function of four variables, $k(x,y,s,t)$ on $[0,1) \times [0,1) \times [0,1) \times [0,1)$ may be approximated with respect to BPFs such as

$$k(x,y,s,t) \simeq \Phi(x,y)^T K \Phi(s,t), \tag{19}$$

where $\Phi(x,y)$ and $\Phi(s,t)$ are 2D-BPFs vectors of dimensional $m_1 m_2$ and $m_3 m_4$, respectively, and K is the $(m_1 m_2) \times (m_3 m_4)$ 2D-BPFs coefficient matrix. For convenience, we put $m_1 = m_3$ and $m_2 = m_4$, so the coefficients $k_{i,j,p,l}$, with $i, p = 1, \ldots, m_1$ and $j, l = 1, \ldots, m_2$ are:

$$k_{i,j,p,l} = (m_1 m_2)^2 \int_0^1 \int_0^1 \int_0^1 \int_0^1 k(x,y,s,t) \phi_{i,j}(x,y) \phi_{p,l}(s,t) dx dy ds dt. \tag{20}$$

Now we expand a function of three variables, $u(x,y,r)$ belongs to $L^2([0,1) \times [0,1) \times [0,1))$ with BPFs as

$$u(x,y,r) \simeq \Phi(x,y)^T U \Phi(r), \tag{21}$$

where $\Phi(x,y)$ is an m_1m_2-component 2D-BPFs vector and $\Phi(r)$ is an m_3-component 1D-BPFs vector, and U is an $(m_1m_2 \times m_3)$ BPFs matrix as

$$U = \begin{pmatrix} u_{1,1,1} & \cdots & u_{1,1,m_3} \\ \vdots & \ddots & \vdots \\ u_{1,m_2,1} & \cdots & u_{1,m_2,m_3} \\ \vdots & \ddots & \vdots \\ u_{m_1,1,1} & \cdots & u_{m_1,1,m_3} \\ \vdots & \ddots & \vdots \\ u_{m_1,m_2,1} & \cdots & u_{m_1,m_2,m_3} \end{pmatrix}, \tag{22}$$

where $u_{i,j,k} = m_1m_2m_3 \int_0^1 \int_0^1 \int_0^1 u(x,y,r)\phi_{i,j}(x,y)\phi_k(r)dxdydr$.

6. System of Integral Equation

In this section, we use 2D-BPFs and 1D-BPFs for solving a special system of linear Fredholm integral equations of the second kind with three variables as follows:

$$\sum_{j=1}^{n} u_j(x,y,r) = f_i(x,y,r) + \sum_{j=1}^{n} \int_0^1 \int_0^1 k_{i,j}(x,y,s,t)u_j(s,t,r)dsdt, \quad i = 1,\dots,n, \tag{23}$$

where u_j, f_i belong to $L^2([0,1) \times [0,1) \times [0,1))$ and $k_{i,j}$'s belong to $L^2([0,1) \times [0,1) \times [0,1) \times [0,1))$. The functions u_j, f_i and $k_{i,j}$ can be approximated with respect to BPFs as:

$$\begin{aligned} f_i(x,y,r) &\simeq \Phi(x,y)^T F_i \Phi(r), \\ u_j(x,y,r) &\simeq \Phi(x,y)^T U_j \Phi(r), \\ k_{i,j}(x,y,s,t) &\simeq \Phi(x,y)^T K_{i,j} \Phi(s,t), \end{aligned} \tag{24}$$

where F_i and U_j are the $m_1m_2 \times m_3$ BPFs coefficients matrices, and K is an $m_1m_2 \times m_1m_2$ BPFs coefficient matrix. By substituting (24) into (23), we have:

$$\sum_{j=1}^{n} \Phi(x,y)^T U_j \Phi(r) \simeq$$

$$\Phi(x,y)^T F_i \Phi(r) + \sum_{j=1}^{n} \int_0^1 \int_0^1 \Phi(x,y)^T K_{i,j} \Phi(s,t) \Phi(s,t)^T U_j \Phi(r) dsdt,$$

$$i = 1,\dots,n,$$

and by replacing $\simeq$ with $=$ yields

$$\sum_{j=1}^{n}(I - h_1 h_2 K_{i,j})U_j = F_i, \quad i = 1, \dots n. \tag{25}$$

Set $L_{i,j} = I - h_1 h_2 K_{i,j}$ then we have following linear system:

$$\begin{pmatrix} L_{1,1} & L_{1,2} & \dots & L_{1,n} \\ L_{2,1} & L_{2,2} & \dots & L_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n,1} & L_{1,2} & \dots & L_{n,n} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix}. \tag{26}$$

By solving the above linear system, we find $U_j, j = 1, \dots, n$ then $u_j(x, y, r) \simeq \Phi(x, y)^T U_j \Phi(r)$ is the estimation of (23).

7. Numerical Examples

Now, we apply the method to two examples and we compare the approximate solutions with the exact solutions (see Tables 1 and 2).

Example 6. Consider the following two-dimensional fuzzy Fredholm integral equation [1]:

$$u(x, y) = f(x, y) + \int_0^1 \int_0^1 x^2 ys\ u(s, t)dsdt, \quad 0 \le x, y \le 1,$$

where

$$f(x, y)(r) = \left(x \sin(y/2)(r^2 + r), x \sin(y/2)(4 - r^3 - r)\right), \quad 0 \le r \le 1.$$

The exact solution is

$$\begin{aligned} \underline{u}(x, y, r) &= \left(x \sin(y/2) - 16/21 \left(\cos(1/2) - 1\right) x^2 y\right) (r^2 + r), \\ \overline{u}(x, y, r) &= \left(x \sin(y/2) - 16/21 \left(\cos(1/2) - 1\right) x^2 y\right) (4 - r^3 - r). \end{aligned}$$

Table 1 shows the exact and approximate solution at $(x, y) = (0.3, 0.6)$ for some variant values of r. For convenience, we put $m_1 = m_2 = m_3 = m$.

r	Exact solution $\underline{u}(x,y,r)$	BPFs method $\underline{u}(x,y,r)$	Exact solution $\overline{u}(x,y,r)$	BPFs method $\overline{u}(x,y,r)$
0	0	0.0009	0.3748	0.3748
0.1	0.0103	0.0110	0.3653	0.3657
0.2	0.0225	0.0228	0.3553	0.3560
0.3	0.0365	0.0363	0.3441	0.3452
0.4	0.0525	0.0516	0.3313	0.3329
0.5	0.0703	0.0723	0.3162	0.3154
0.6	0.0899	0.0914	0.2983	0.2979
0.7	0.1115	0.1122	0.2770	0.2773
0.8	0.1349	0.1348	0.2518	0.2530
0.9	0.1602	0.1591	0.2221	0.2245

Table 1: Numerical result for Example 6 with $m = 52$

Example 7. Consider the following two-dimensional fuzzy Fredholm integral equation:

$$u(x,y) = f(x,y) + \int_0^1 \int_0^1 -\frac{7}{6}\pi xysin(\pi s)\ u(s,t)dsdt, \quad 0 \le x,y \le 1,$$

where

$$\underline{f}(x,y,r) = \pi xy(\frac{13}{15}(r^2+r) + \frac{2}{15}(4-r^3-r)),$$
$$\overline{f}(x,y,r) = \pi xy(\frac{2}{15}(r^2+r) + \frac{13}{15}(4-r^3-r)).$$

The exact solution is

$$\underline{u}(x,y,r) = \frac{\pi}{25}xy(\frac{268}{19}r^3 + \frac{568}{19}r^2 + 44r - \frac{1072}{19}),$$
$$\overline{u}(x,y,r) = -\frac{\pi}{25}xy(\frac{568}{19}r^3 + \frac{268}{19}r^2 + 44r - \frac{2272}{19}).$$

Tabe 2 shows the exact and approximate solution at $(x,y) = (0.3, 0.6)$ for some variant values of r.

r	Exact solution $\underline{u}(x,y,r)$	BPFs method $\underline{u}(x,y,r)$	Exact solution $\overline{u}(x,y,r)$	BPFs method $\overline{u}(x,y,r)$
0	-1.2762	-1.2715	2.7048	2.7039
0.1	-1.1696	-1.1671	2.6014	2.6030
0.2	-1.0476	-1.0482	2.4876	2.4922
0.3	-0.9082	-0.9129	2.3593	2.3677
0.4	-0.7495	-0.7596	2.2124	2.2258
0.5	-0.5697	-0.5556	2.0429	2.0335
0.6	-0.3667	-0.3572	1.8467	1.8413
0.7	-0.1388	-0.1352	1.6199	1.6202
0.8	0.1161	0.1121	1.3582	1.3663
0.9	0.3998	0.3865	1.0577	1.0760

Table 2: Numerical result for Example 7 with $m = 62$

8. Conclusion

In this work we present a numerical method based on block pulse functions. We can see that, solving a linear two-dimensional fuzzy Fredholm integral equation of the second kind is converted to solving a system of linear Fredholm integral equations of the second kind with three variables. By using BPFs we transform solving this system to solve a linear equations system. By two examples we tried to find approximate solutions for which the results seem acceptable. The benefit of this method is low cost of setting up the equations without applying any projection method such as Galerkin, collocation, etc.

References

[1] A. Rivaz, F. Yousefi, Numerical methods to solve two-dimensional fuzzy Fredholm integral equations of the second kind, *Submitted*.

[2] E. Babolian, Z. Masouri, Direct method to solve voltrra integral equation of the first kind using operational matrix with block pulse functions, *J. Comput. Appl. Math.*, **220** (2008), 51-57.

[3] K. Maleknejad, S. Sohrabi, B. Baranji, Application of 2D-BPFs to non-linear integral equations, *Commun. Nonlinear Sci. Nnmer. Simulat.*, **15** (2010), 527-532.

[4] M. Friedman, M. Ma and A. Kandel, Numerical solution of fuzzy differential and integral equations, *Fuzzy Sets Systems*, **106** (1999), 35-48.

[5] M. Ghanbari, R. Toushmalni, E. Kamrani, Numerical solution of linear Fredholm fuzzy integral equation of the second kind by block pulse functions, *Aust. J. Basic Appl. Sci.*, **3(3)** (2009), 2637-2642.

[6] M. Matloka, On fuzzy integrals, In: *Proc. 2-nd Polish Symp.*, *On Interval and Fuzzy Mathematics*, Politechnika Poznansk (1987), 167-170.

[7] M. L. Puri, D. Ralescu, Fuzzy random variables, *J. Math. Anal. Appl.*, **114** (1986), 409-422.

[8] O. Kaleva, Fuzzy differential equations, *Fuzzy Sets and Systems*, **24** (1987), 301-317.

[9] R. Goetschel, W. Voxman, Elementary calculus, *Fuzzy Sets and Systems*, **18** (1986), 31-43.

[10] S. Nanda, On integration of fuzzy mappings, *Fuzzy Sets and Systems*, **32** (1989), 95-101.