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ON STARLIKENESS OF CONFLUENT HYPERGEOMETRIC FUNCTIONS

Kazuo Kuroki[1], Shigeyoshi Owa[2 §], Ismet Yildiz[3]
[1,2]Department of Mathematics
Kinki University
Higashi-Osaka, Osaka, 577-8502, JAPAN
[1]e-mail: freedom@sakai.zaq.ne.jp
[2]e-mail: shige21@ican.zaq.ne.jp
[3]Department of Mathematics
Duzce University
Konuralp Yerleskesi, Duzce 81620, TURKEY
e-mail: ismetyildiz@duzce.edu.tr

Abstract: The object of the present paper is to discuss some conditions for confluent hypergeometric functions to be starlike of order α in the open unit disk.

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1. Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ in $\mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

for some real α $(0 \leqq \alpha < 1)$.

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§Correspondence author

The confluent hypergeometric function $F(z)$ is defined by

$$F(z) \equiv {}_1F_1(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{(1)_n} \tag{1.1}$$

with $a \in \mathbb{C}$, $c \in \mathbb{C}$ and $c \neq 0, -1, -2, \cdots$, where $(a)_n$ denotes the Pochhammer symbol

$$(a)_n = \begin{cases} a(a+1)(a+2)\cdots(a+n-1) & (n \neq 0) \\ 1 \qquad (a \neq 0, n = 0). \end{cases}$$

We note that the confluent hypergeometric function $F(z)$ satisfies the following Kummer hypergeometric differential equation

$$zF''(z) + (c-z)F'(z) - aF(z) = 0.$$

2. Some Lemmas

For univalency of the confluent hypergeometric functions, Miller and Mocanu [2] have given the following results.

Lemma 2.1. *Let $a \in \mathbb{R}$ and $c \in \mathbb{R}$. If one of the following conditions*

(i) $a > 0$ and $c \geqq a$

or

(ii) $a \leqq 0$ and $c \geqq 1 + \sqrt{1+a^2}$

is satisfied, then

$$\operatorname{Re}\{{}_1F_1(a, c; z)\} > 0 \qquad (z \in \mathbb{U}).$$

Lemma 2.2. *Let $a \in \mathbb{R}$ and $c \in \mathbb{R}$. If one of the following conditions*

(i) $a > -1$ and $c \geqq a$

or

(ii) $a \leqq -1$ and $c \geqq \sqrt{1+(1+a)^2}$

is satisfied, then

$$\operatorname{Re}\left\{\frac{c}{a}{}_1F_1{}'(a, c; z)\right\} > 0 \qquad (z \in \mathbb{U}).$$

Therefore, ${}_1F_1(a, c; z)$ *is univalent in* $\mathbb{U}$.

From the above lemmas, we have the following problems.

Problem 1. Find $a \in \mathbb{C}$ and $c \in \mathbb{C}$ such that ${}_1F_1(a, c; z) \neq 0$ for $z \in \mathbb{U}$.

Problem 2. Find $a \in \mathbb{C}$ and $c \in \mathbb{C}$ such that ${}_1F_1{}'(a, c; z) \neq 0$ for $z \in \mathbb{U}$.

To study the starlikeness of the confluent hypergeometric functions $F(z)$ defined by (1.1), we need the following

Lemma 2.3. (see [1]) *Let* $\mathbb{E}$ *be a set in the complex plane* $\mathbb{C}$, *and let a function* $H : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ *satisfy* $H(is, t; z) \notin \mathbb{E}$ *for all* $z \in \mathbb{U}$ *and for all real* s *and* t *satisfying* $t \leqq -\frac{1+s^2}{2}$. *If* $p(z)$ *is analytic in* $\mathbb{U}$ *with* $p(0) = 1$ *and* $H\big(p(z), zp'(z); z\big) \in \mathbb{E}$ *for all* $z \in \mathbb{U}$, *then* $\operatorname{Re} p(z) > 0 \quad (z \in \mathbb{U})$.

Let us mention also that Owa and Srivastava [3] and Srivastava and Owa [4] have discussed some interesting results for univalency and starlikeness of generalized hypergeometric functions.

3. Starlikeness

Our result for the confluent hypergeometric functions is given by the following theorem.

Theorem 3.1. *Let* $a = a_1 + ia_2$, $c = c_1 + ic_2$ *with* $c \neq 0, -1, -2, \cdots$, *and* $a_2 \geqq \frac{1-\alpha}{3-2\alpha} c_2$. *Let* ${}_1F_1(a, c; z) \neq 0 \quad (z \in \mathbb{U})$. *If one of the following conditions*

(i)

$$a_1 > \frac{2(1-\alpha)(2-\alpha)}{3-2\alpha}$$

and

$$c_2{}^2 - 2c_2 - (3-2\alpha)(2c_1 + 2\alpha - 3) \leqq -\frac{2(3-2\alpha)}{1-\alpha}(a_1 + a_2 + \alpha - 1),$$

(ii)

$$\frac{2(1-\alpha)^2}{3-2\alpha} < a_1 \leqq \frac{2(1-\alpha)(2-\alpha)}{3-2\alpha}$$

and

$$(1-c_2)^2-(3-2\alpha)(2c_1+2\alpha-3) \leqq -\frac{3-2\alpha}{1-\alpha}\left\{\frac{(3-2\alpha)(1-\alpha-a_1)^2}{1-\alpha}+2a_2\right\},$$

or

(iii)

$$a_1 \leqq \frac{2(1-\alpha)^2}{3-2\alpha}$$

and

$$c_2{}^2-2c_2-(3-2\alpha)(2c_1+2\alpha-3) \leqq \frac{2(3-2\alpha)}{1-\alpha}(a_1-a_2+\alpha-1)$$

is satisfied for some α $(0 \leqq \alpha < 1)$, *then* $z\,{}_1F_1(a,c;z)$ *is starlike of order* α *in* $\mathbb{U}$.

Proof. Let us denote ${}_1F_1(a,c;z)$ by $F(z)$. Then the function $F(z)$ satisfies

$$\frac{z^2F''(z)}{F(z)}+(c-z)\frac{zF'(z)}{F(z)}-az=0.$$

Letting $g(z) \equiv zF(z)$ and

$$p(z)=\frac{\dfrac{zg'(z)}{g(z)}-\alpha}{1-\alpha} \qquad \big(p(0)=1\big),$$

we see that

$$\begin{aligned}&\frac{z^2F''(z)}{F(z)}+(c-z)\frac{zF'(z)}{F(z)}-az\\ &=(1-\alpha)\Big\{zp'(z)+(1-\alpha)\big(p(z)\big)^2\\ &+(c-3+2\alpha-z)p(z)+\frac{1-\alpha-a}{1-\alpha}z+(2-\alpha-c)\Big\}=0.\end{aligned}$$

If we define $\mathbb{E}=\{0\}$ and

$$H(w_1,w_2;z)=w_2+(1-\alpha)w_1^2+(c-3+2\alpha-z)w_1+\frac{1-\alpha-a}{1-\alpha}z+(2-\alpha-c),$$

then the function $H(w_1,w_2;z)$ satisfies

$$H\big(p(z),zp'(z);z\big)=0\in\mathbb{E}$$

for all $z \in \mathbb{U}$. Note that

$$\begin{aligned}
&\mathrm{Re}\{H(is,t;z)\}\\
&= t-(1-\alpha)s^2-(c_2-y)s+\frac{1-\alpha-a_1}{1-\alpha}x+\frac{a_2}{1-\alpha}y+(2-\alpha-c_1)\\
&\leqq -\frac{1}{2}\left\{(3-2\alpha)s^2+2(c_2-y)s-\frac{2(1-\alpha-a_1)}{1-\alpha}x\right.\\
&\qquad\qquad\left.-\frac{2a_2}{1-\alpha}y+2c_1+2\alpha-3\right\}
\end{aligned}$$

for $z=x+iy$, and for all s, t, such that $t \leqq -\frac{1+s^2}{2}$.

Let us define the function $Q(s)$ by

$$Q(s)=(3-2\alpha)s^2+2(c_2-y)s-\frac{2(1-\alpha-a_1)}{1-\alpha}x-\frac{2a_2}{1-\alpha}y+2c_1+2\alpha-3.$$

Then the discrimination Δ of $Q(s)$ satisfies

$$\Delta=(y-c_2)^2-(3-2\alpha)\left(-\frac{2(1-\alpha-a_1)}{1-\alpha}x-\frac{2a_2}{1-\alpha}y+2c_1+2\alpha-3\right)$$

$$\begin{aligned}
&< -x^2+1+{c_2}^2-(3-2\alpha)\left\{-\frac{2(1-\alpha-a_1)}{1-\alpha}x+2\left(\frac{c_2}{3-2\alpha}-\frac{a_2}{1-\alpha}\right)\right.\\
&\qquad\qquad \left.+\,2c_1+2\alpha-3\right\}.
\end{aligned}$$

If we define the function $h(x)$ by

$$\begin{aligned}
h(x)&=-x^2+1+{c_2}^2-(3-2\alpha)\left\{-\frac{2(1-\alpha-a_1)}{1-\alpha}x\right.\\
&\qquad\left.+2\left(\frac{c_2}{3-2\alpha}-\frac{a_2}{1-\alpha}\right)+2c_1+2\alpha-3\right\},
\end{aligned}$$

then

$$h'(x)=-2(x-x_0),$$

where

$$x_0=\frac{(3-2\alpha)(1-\alpha-a_1)}{1-\alpha}.$$

Therefore, the conditions (i), (ii), (iii) of the theorem lead to:

(*i*) if $x_0 < -1$, equivalently if

$$a_1 > \frac{2(1-\alpha)(2-\alpha)}{3-2\alpha},$$

then $h(x) < h(-1) \leqq 0$.

(*ii*) if $-1 \leqq x_0 < 1$, equivalently if

$$\frac{2(1-\alpha)^2}{3-2\alpha} < a_1 \leqq \frac{2(1-\alpha)(2-\alpha)}{3-2\alpha},$$

then $h(x) < h(x_0) \leqq 0$.

(*iii*) if $x_0 \geqq 1$, equivalently if

$$a_1 \leqq \frac{2(1-\alpha)^2}{3-2\alpha},$$

then $h(x) < h(1) \leqq 0$.

Thus we have $\Delta < 0$ with the conditions (*i*), (*ii*), (*iii*) of the theorem. This implies that $Q(s) > 0$, that is that

$$\mathrm{Re}\big\{H(is,t;z)\big\} < 0 \qquad (z \in \mathbb{U}).$$

Therefore, we conclude that $H(is,t;z) \notin \mathbb{E}$.

Finally, applying Lemma 2.3, we obtain $\mathrm{Re}\, p(z) > 0 \quad (z \in \mathbb{U})$ which is equivalent to $z_1F_1(a,c;z)$ is starlike of order α in $\mathbb{U}$. □

Remark 3.2. (i) Taking $a = 2+i$, $c = 5+3i$, $\alpha = 0$ in Theorem 3.1, we have $z_1F_1(2+i, 5+3i; z)$ is starlike for $|z| < r_0$, where r_0 is the smallest positive real number such that ${}_1F_1(2+i, 5+3i; z) \neq 0 \quad (|z| \leqq r_0)$.

(ii) If we take $a = 1+i$, $c = 8+4i$, $\alpha = \frac{1}{2}$ in Theorem 3.1, then $z_1F_1(1+i, 8+4i; z)$ is starlike of order $\frac{1}{2}$ for $|z| < r_0$, where r_0 is the smallest positive real number such that ${}_1F_1(1+i, 8+4i; z) \neq 0 \quad (|z| \leqq r_0)$.

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