

**A QR-DECOMPOSITION OF THE MEAN VALUE  
MATRIX OF THE COEFFICIENT MATRIX FOR  
SOLVING THE FULLY FUZZY LINEAR SYSTEM**

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**Abstract:** We give a suitable decomposition of the mean value of the coefficient matrix of the square or rectangular Fully Fuzzy Linear System (FFLS) to design a simple algorithm for solving FFLSs. The mentioned method can solve FFLS in a smaller computing process. We will illustrate our method by solving some examples.

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## 1. Introduction

One major application of the fuzzy number arithmetic is treating linear systems whose parameters are all or partially represented by fuzzy numbers. The term fuzzy matrix, which is the most important concept in this paper, has various meanings. For the definition of a fuzzy matrix we follow the definition of Dubois and Prade, i.e. a matrix with fuzzy numbers as its elements [5].

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This class of fuzzy matrices consists of applicable matrices, which can model uncertain aspects and the studies on them are too limited. Some of the most interesting works on these matrices can be seen in [2-5, 7-10]. A general model for solving a fuzzy linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy vector, first proposed by Friedman et al. [6]. Friedman and his colleagues used the embedding method and replaced the original fuzzy linear system by a crisp linear system and then they solved it. A review of some methods for solving these systems can be found in [4]. In addition, another important kind of fuzzy linear systems is including fuzzy numbers in all parameters and these are named fully fuzzy linear systems (see in [3, 4, 10]). Nevertheless, there is just a few computational methods for solving a fully fuzzy linear system until now. For example, recently Dehghan and his colleagues in [3] and [4] proposed two numerical methods for solving this kind of systems. Recently, we presented a new method to solve fully fuzzy linear systems based on LU decomposition of matrices (see [10]). In this paper we intend to design a method for solving  $\tilde{A} \otimes \tilde{x} = \tilde{b}$ , where  $\tilde{A}$  is a fuzzy matrix and  $\tilde{x}$  and  $\tilde{b}$  are fuzzy vectors with appropriate sizes. The paper is organized in 5 sections.

In Section 2, we first give some basic concepts of the fuzzy sets theory and then define a general fully fuzzy linear system of equations. A numerical method for computing the solution of FFLS is designed in Section 3. Numerical examples are given in Section 4 to illustrate our method. Finally, we conclude in Section 5.

## 2. Preliminaries

In this section, we give some necessary background and notions of fuzzy sets theory (taken from [5, 7]).

**Definition 2.1.** A fuzzy subset  $\tilde{A}$  of  $\mathbb{R}$  is defined by its membership function

$$\mu_{\tilde{A}} : \mathbb{R} \rightarrow [0, 1],$$

which assigns a real number  $\mu_{\tilde{A}}$  in the interval  $[0, 1]$ , to each element  $x \in \mathbb{R}$ , where the value of  $\mu_{\tilde{A}}$  at  $x$  shows the grade of membership of  $x$  in  $\tilde{A}$ . Indeed, a fuzzy subset  $\tilde{A}$  can be characterized as a set of ordered pairs of element  $x$  and grade  $\mu_{\tilde{A}}$  and is often written

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in \mathbb{R}\}.$$

**Definition 2.2.** A fuzzy set with the following membership function is named a triangular fuzzy number, and in this paper we will use these fuzzy numbers,

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \frac{m-x}{\alpha}, & m - \alpha \leq x < m, \alpha > 0, \\ 1 - \frac{x-m}{\beta}, & m \leq x \leq m + \beta, \beta > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (2.1)$$

**Definition 2.3.** A fuzzy number  $\tilde{A}$  is called positive (negative), denoted by  $\tilde{A} > 0$  ( $\tilde{A} < 0$ ), if its membership function  $\mu_{\tilde{A}}(x)$  satisfies  $\mu_{\tilde{A}}(x) = 0, \forall x \leq 0$  ( $\forall x \geq 0$ ).

Using its mean value and left and right spreads, and shape functions, such a triangular fuzzy number  $\tilde{A}$  is symbolically written

$$\tilde{A} = (m, \alpha, \beta).$$

Clearly,  $\tilde{A} = (m, \alpha, \beta)$  is positive, if and only if  $m - \alpha \geq 0$ .

**Remark 2.1.** We consider  $\tilde{0} = (0, 0, 0)$  as zero fuzzy number.

**Remark 2.2.** We denote the set of all triangular fuzzy numbers by  $F(\mathbb{R})$ .

**Definition 2.4.** (Equality in fuzzy numbers). Two triangular fuzzy numbers  $M = (m, \alpha, \beta)$  and  $N = (n, \gamma, \delta)$  are said to be equal, if and only if  $m = n, \alpha = \gamma$  and  $\beta = \delta$ .

**Definition 2.5.** For two triangular fuzzy numbers  $M = (m, \alpha, \beta)$  and  $N = (n, \gamma, \delta)$  the formula for the extended addition becomes:

$$(m, \alpha, \beta) \oplus (n, \gamma, \delta) = (m + n, \alpha + \gamma, \beta + \delta). \quad (2.2)$$

The formula for the extended opposite becomes:

$$-M = -(m, \alpha, \beta) = (-m, \beta, \alpha). \quad (2.3)$$

Let  $M = (m, \alpha, \beta)$  and  $N = (n, \gamma, \delta)$  be two triangular fuzzy numbers, respectively,

$$M \ominus N = (m, \alpha, \beta) \ominus (n, \gamma, \delta) = (m - n, \alpha + \delta, \beta + \gamma). \quad (2.4)$$

The approximate formulas for the extended multiplication of two triangular fuzzy numbers can be summarized as follows (given in [5]):

If  $M > 0$  and  $N > 0$ , then

$$(m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, m\gamma + n\alpha, m\delta + n\beta). \quad (2.5)$$

For scalar multiplication:

$$\lambda \otimes M = \lambda \otimes (m, \alpha, \beta) = \begin{cases} (\lambda m, \lambda \alpha, \lambda \beta), & \lambda \geq 0, \\ (\lambda m, -\lambda \beta, -\lambda \alpha), & \lambda < 0. \end{cases} \quad (2.6)$$

**Definition 2.6.** A matrix  $\tilde{A} = (\tilde{a}_{ij})$  is called a fuzzy matrix, if each element of  $\tilde{A}$  is a fuzzy number.

A fuzzy matrix  $\tilde{A}$  will be positive and denoted by  $\tilde{A} > \tilde{0}$ , if each element of  $\tilde{A}$  be positive. We may represent  $n \times n$  fuzzy matrix  $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ , such that  $\tilde{a}_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})$ , with the new notation  $\tilde{A} = (A, M, N)$ , where  $A = (a_{ij})$ ,  $M = (\alpha_{ij})$  and  $N = (\beta_{ij})$  are three  $n \times n$  crisp matrices.

**Definition 2.7.** A square fuzzy matrix  $\tilde{A} = (\tilde{a}_{ij})$  will be an upper triangular matrix, if

$$\tilde{a}_{ij} = \tilde{0} = (0, 0, 0), \quad \forall i > j.$$

**Definition 2.8.** Consider the  $n \times n$  fuzzy linear system of equations [3, 10]:

$$\left\{ \begin{array}{l} (\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = \tilde{b}_1, \\ (\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = \tilde{b}_2, \\ \vdots \\ (\tilde{a}_{m1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{m2} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{mn} \otimes \tilde{x}_n) = \tilde{b}_m. \end{array} \right. \quad (2.7)$$

The matrix form of the above equations is

$$\tilde{A} \otimes \tilde{x} = \tilde{b},$$

where the coefficient matrix  $\tilde{A} = (\tilde{a}_{ij})$ ,  $1 \leq i \leq m$ ,  $1 \leq j \leq n$  is an  $m \times n$  fuzzy matrix such that  $\tilde{a}_{ij} \in F(\mathbb{R})$  and  $\tilde{x}_j, \tilde{b}_i \in F(\mathbb{R})$ , for all  $i = 1, \dots, m$  and  $j = 1, \dots, n$ . This system is called a Fully Fuzzy Linear System (FFLS).

In this paper we are going to obtain a positive solution of FFLS  $\tilde{A} \otimes \tilde{x} = \tilde{b}$ , where  $\tilde{A} = (A, M, N) > \tilde{0}$ ,  $\tilde{b} = (b, h, g) > \tilde{0}$  and  $\tilde{x} = (x, y, z) > \tilde{0}$ . So we have

$$(A, M, N) \otimes (x, y, z) = (b, h, g). \quad (2.8)$$

Then by using Eq.(2.5) we have

$$(Ax, Ay + Mx, Az + Nx) = (b, h, g). \quad (2.9)$$

Therefore, Definition 2.4 concludes that

$$\begin{cases} Ax = b, \\ Ay + Mx = h, \\ Az + Nx = g. \end{cases} \quad (2.10)$$

So, by assuming that  $A$  be a nonsingular matrix, we have

$$\begin{cases} Ax = b & \Rightarrow x = A^{-1}b, \\ Ay = h - Mx & \Rightarrow y = A^{-1}(h - Mx), \\ Az = g - Nx & \Rightarrow z = A^{-1}(g - Nx). \end{cases}$$

### 3. New Method for Solving FFLS

In this section we present a new method which is completely suitable for rectangular fully fuzzy linear system of equations. Hence we first give a theorem concerning the QR decomposition of matrices and then focus on solving these systems.

**Theorem 3.1.** *If  $A$  is an  $m \times k$  matrix with full column rank, then  $A$  can be factored as  $A = QR$  where  $Q$  is an  $m \times k$  matrix whose column vectors form an orthonormal basis for the column space of  $A$  and  $R$  is a  $k \times k$  invertible upper triangular matrix (see [1]).*

Now consider the fully fuzzy linear system as defined in Eq.(2.8). Assuming that in the fully fuzzy coefficient matrix  $\tilde{A} = (A, M, N)$ ,  $A$  is a full rank crisp matrix and its QR-decomposition is  $A = Q_1 R_1$ , then we have:

$$\tilde{A} = (A, M, N) = (Q_1, Q_2, Q_3) \otimes (R_1, 0, 0) = (Q_1 R_1, Q_2 R_1, Q_3 R_1). \quad (3.1)$$

Therefore

$$\begin{cases} Q_1 R_1 = A & \Rightarrow R_1 = Q_1^T A \\ Q_2 R_1 = M & \Rightarrow Q_2 = M R_1^{-1} \\ Q_3 R_1 = N & \Rightarrow Q_3 = N R_1^{-1} \end{cases}, \quad (3.2)$$

where the matrix  $Q_1$  is an orthonormal crisp matrix and the matrix  $R_1$  is an upper triangular crisp matrix. Hence, Eq.(2.8) and Eq.(3.1) yield that

$$(Q_1 R_1, Q_2 R_1, Q_3 R_1) \otimes (x, y, z) = (b, g, h),$$

or equivalently,

$$(Q_1 R_1 x, Q_2 R_1 x + Q_1 R_1 y, Q_3 R_1 x + Q_1 R_1 z) = (b, g, h).$$

Finally, the solution of the original fully fuzzy linear system can be achieved as follows:

$$\begin{cases} Q_1 R_1 x = b & \Rightarrow x = R_1^{-1} Q_1^T b \\ Q_2 R_1 x + Q_1 R_1 y = g & \Rightarrow y = R_1^{-1} Q_1^T (g - Mx) \\ Q_3 R_1 x + Q_1 R_1 z = h & \Rightarrow z = R_1^{-1} Q_1^T (h - Nx) \end{cases} \quad (3.3)$$

#### 4. Numerical Examples

**Example 4.1.** Consider the following FFSL [3]:

$$\begin{pmatrix} (5, 1, 1) & (6, 1, 2) \\ (7, 1, 0) & (4, 0, 1) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} (50, 10, 17) \\ (48, 5, 7) \end{pmatrix}.$$

First we obtain QR-decomposition for matrix  $A$ ,

$$A = Q_1 R_1,$$

$$\begin{pmatrix} 5 & 6 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} -0.5812 & -0.8137 \\ -0.8137 & 0.5812 \end{pmatrix} \begin{pmatrix} -8.6023 & -6.7423 \\ 0 & -2.5574 \end{pmatrix}.$$

So using Eq. (3.2), we obtain the matrices  $Q_2, Q_3$ :

$$Q_2 = M R_1^{-1} = \begin{pmatrix} -0.1162 & -0.0845 \\ -0.1162 & 0.3064 \end{pmatrix},$$

$$Q_3 = N R_1^{-1} = \begin{pmatrix} -0.1162 & -0.4755 \\ 0 & -0.3910 \end{pmatrix}.$$

By using Eq.(3.3), we have

$$\tilde{x} = (4, \frac{1}{11}, 0), \quad \tilde{y} = (5, \frac{1}{11}, \frac{1}{2}).$$

**Example 4.2.** Consider the following FFSL (taken from [4]):

$$\begin{pmatrix} (19, 1, 1) & (12, 1.5, 1.5) & (6, 0.5, 0.2) \\ (2, 0.1, 0.1) & (4, 0.1, 0.4) & (1.5, 0.2, 0.2) \\ (2, 0.1, 0.2) & (2, 0.1, 0.3) & (4.5, 0.1, 0.1) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} (1897, 427.7, 536.2) \\ (434.5, 76.2, 109.3) \\ (535.5, 88.3, 131.9) \end{pmatrix}.$$

First we obtain QR-decomposition for matrix  $A$  as follows:

$$A = \begin{pmatrix} 19 & 12 & 6 \\ 2 & 4 & 1.5 \\ 2 & 2 & 4.5 \end{pmatrix} \\ = \begin{pmatrix} -0.9891 & 0.1272 & -0.0740 \\ -0.1041 & -0.9601 & -0.2592 \\ -0.1041 & -0.2487 & 0.9629 \end{pmatrix} \begin{pmatrix} -19.2093 & 0.1272 & -0.0740 \\ 0 & -2.8111 & -1.7959 \\ 0 & 0 & 3.5000 \end{pmatrix}.$$

So using Eq. (3.2), we obtain the matrices  $Q_2, Q_3$ :

$$Q_2 = MR_1^{-1} = \begin{pmatrix} -0.0520 & -0.3022 & 0.1097 \\ -0.0052 & -0.0124 & 0.0410 \\ -0.0052 & -0.0124 & 0.0124 \end{pmatrix}, \\ Q_3 = NR_1^{-1} = \begin{pmatrix} -0.0520 & -0.3022 & -0.1955 \\ -0.0052 & -0.1191 & -0.0137 \\ -0.0104 & -0.0604 & -0.0219 \end{pmatrix}.$$

Therefore, Eq.(3.3) concludes that

$$\tilde{x} = (36.9999, 7, 13.3015),$$

$$\tilde{y} = (61.9999, 5.5, 4.5793),$$

$$\tilde{z} = (74.9999, 10.1999, 13.9195).$$

## 5. Conclusion

Using the QR-decomposition, we present a new method for solving FFLS. In fact, initially we obtain the QR-decomposition for the mean value matrix  $A$  such that  $A = Q_1 R_1$  and then using  $R_1^{-1}$  and  $Q_1^T$ , we compute the solution of the original rectangular fully fuzzy linear system of equations. We see the mentioned method allows to obtain the solution with a simple computing process.

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