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COMPACTNESS OF THE WEIGHTED MEAN OPERATOR MATRIX ON WEIGHTED HARDY SPACES

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Abstract: In this paper we characterize the compactness of the weighted mean operator matrix on Banach spaces $H^p(\beta)$.

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1. Introduction

Let X and Y be normed linear spaces. Suppose T is a linear operator with domain X and range in Y. We say that T is compact, if the image $T(B)$ of closed unit ball $B = \{x : \|x\| \leq 1\} \subseteq X$ is relatively compact, that is, $cl[T(B)]$ is compact in Y.

Let $\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$ and $1 < p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that $\|f\|^p = \|f\|_\beta{}^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty$. The notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ shall be used whether or not the series converges for any value of z. These are called formal power series and the set of such series is denoted by $H^p(\beta)$. Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_k$ is a basis such that $\|f_k\| = \beta(k)$. Recall that $H^p(\beta)$ is a reflexive Banach space with norm $\|.\|_\beta$ and the dual of $H^p(\beta)$ is

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$H^q(\beta^{\frac{p}{q}})$ where $\frac{1}{p}+\frac{1}{q}=1$ and $\beta^{\frac{p}{q}}=\{\beta(n)^{\frac{p}{q}}\}$.

Let $\{a_n\}$ be a sequence of positive numbers, and let $A_n=\sum_{i=0}^{n} a_i\beta(i)^p$. Define the weighted mean operator matrix $A=[a_{nk}]_{n,k}$ on $H^p(\beta)$ by

$$a_{nk}=\begin{cases}\frac{a_k\beta(n)^p}{A_n} & 0\le k\le n\\ 0 & k>n\end{cases}.$$

Note that if $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n\in H^p(\beta)$, then

$$Af(z)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}(\frac{a_k\hat{f}(k)\beta(n)^p}{A_n})z^n.$$

In this paper we investigate the compactness of A. For some sources on these topics, see [1-10].

2. Main Results

We will find conditions under which the weighted mean operator matrix A is compact.

Lemma 2.1. *Let $a_{ij}\ge 0$ for $n\le i\le m, 0\le j\le m$. Then*

$$\sum_{i=n}^{m}(\sum_{j=0}^{i}a_{ij})^p\le(\sum_{j=0}^{m}(\sum_{i=n}^{m}a_{ij}^p)^{\frac{1}{p}}+\sum_{j=n+1}^{m}(\sum_{i=j}^{m}a_{ij}^p)^{\frac{1}{p}})^p.$$

Proof. First define

$$b_{ij}=\begin{cases}a_{ij} & 0\le j\le i\le m\\ 0 & n\le i\le j\le m\end{cases}.$$

The Hölder's inequality yields

$$\sum_{i=n}^{m}b_{ij}(\sum_{j=0}^{m}b_{ij})^{\frac{p}{q}}\le(\sum_{i=n}^{m}b_{ij}^p)^{\frac{1}{p}}(\sum_{i=n}^{m}(\sum_{j=0}^{m}b_{ij})^p)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$. Thus $\sum_{i=n}^{m}(\sum_{j=0}^{m}b_{ij})^p\le(\sum_{j=0}^{m}(\sum_{i=n}^{m}b_{ij}^p)^{\frac{1}{p}})^p$. Therefore we get

$$\sum_{i=n}^{m}(\sum_{j=0}^{i}a_{ij})^p \leq \sum_{i=n}^{m}(\sum_{j=0}^{m}b_{ij})^p$$
$$\leq (\sum_{j=0}^{n}(\sum_{i=n}^{m}a_{ij}^p)^{\frac{1}{p}} + \sum_{j=n+1}^{m}(\sum_{i=j}^{m}a_{ij}^p)^{\frac{1}{p}})^p.$$

This completes the proof. □

Theorem 2.2. *Let the weighted mean matrix operator A be bounded on $H^p(\beta)$. Also, suppose that*

$$\lim_{m\longrightarrow\infty}\sum_{k=0}^{m}(\sum_{n=m}^{\infty}(\frac{a_k\beta(n)^{p+1}}{\beta(k)A_n})^p)^{\frac{q}{p}} = 0,$$

and

$$\lim_{m\longrightarrow\infty}\sum_{k=m+1}^{\infty}(\sum_{n=k}^{\infty}(\frac{a_k\beta(n)^{p+1}}{\beta(k)A_n})^p)^{\frac{q}{p}} = 0,$$

where $\frac{1}{p}+\frac{1}{q}=1$. Then A is a compact operator on $H^p(\beta)$.

Proof. Suppose that $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n \in B_{H^p(\beta)}$. If $\epsilon>0$, then there exists $m_0\in\mathbb{N}$, such that for $m\geq m_0$ we have

$$\sum_{k=0}^{m}(\sum_{n=m}^{\infty}(\frac{a_k\beta(n)^{p+1}}{\beta(k)A_n})^p)^{\frac{q}{p}} < (\frac{\epsilon}{2})^q,$$
$$\sum_{k=m+1}^{\infty}(\sum_{n=k}^{\infty}(\frac{a_k\beta(n)^{p+1}}{\beta(k)A_n})^p)^{\frac{q}{p}} < (\frac{\epsilon}{2})^q.$$

Define $S=A(B_{H^p(\beta)})$ and note that S is a subset of $H^p(\beta)$. By Lemma 2.1 we get

$$(\sum_{n=m}^{\infty}|\sum_{k=0}^{n}\frac{a_k\beta(n)^p\hat{f}(k)}{A_n}|^p\beta(n)^p)^{\frac{1}{p}} \leq (\sum_{n=m}^{\infty}(\sum_{k=0}^{n}\frac{a_k\beta(n)^{p+1}|\hat{f}(k)|\beta(k)}{A_n\beta(k)})^p)^{\frac{1}{p}}$$
$$\leq \sum_{k=0}^{m}(\sum_{n=m}^{\infty}(\frac{a_k\beta(n)^{p+1}}{\beta(k)A_n})^p)^{\frac{1}{p}}|\hat{f}(k)|\beta(k)$$
$$+\sum_{k=m+1}^{\infty}(\sum_{n=k}^{\infty}(\frac{a_k\beta(n)^{p+1}}{\beta(k)A_n})^p)^{\frac{1}{p}}|\hat{f}(k)|\beta(k).$$

Put

$$F_1 = \sum_{k=0}^{m} (\sum_{n=m}^{\infty} (\frac{a_k \beta(n)^{p+1}}{\beta(k) A_n})^p)^{\frac{1}{p}} |\hat{f}(k)| \beta(k).$$

By the Hölder inequality we get

$$F_1 \leq (\sum_{k=0}^{m} (\sum_{n=m}^{\infty} (\frac{a_k \beta(n)^{p+1}}{\beta(k) A_n})^p)^{\frac{q}{p}})^{\frac{1}{q}} (\sum_{k=0}^{m} |\hat{f}(k)|^p \beta(k)^p)^{\frac{1}{p}} \leq \frac{\epsilon}{2} \|f\|_\beta.$$

By the similar method, if we define

$$F_2 = \sum_{k=m+1}^{\infty} (\sum_{n=k}^{\infty} (\frac{a_k \beta(n)^{p+1}}{\beta(k) A_n})^p)^{\frac{1}{p}} |\hat{f}(k)| \beta(k),$$

then we can see that

$$F_2 \leq (\sum_{k=m+1}^{\infty} (\sum_{n=k}^{\infty} (\frac{a_k \beta(n)^{p+1}}{\beta(k) A_n})^p)^{\frac{q}{p}})^{\frac{1}{q}} (\sum_{k=m+1}^{\infty} |\hat{f}(k)|^p \beta(k)^p)^{\frac{1}{p}} \leq \frac{\epsilon}{2} \|f\|.$$

So for $m \geq m_0$, we obtain

$$\sum_{n=m}^{\infty} |\hat{Af}(n)|^p \beta(n)^p \leq \sum_{n=m}^{\infty} |\sum_{k=0}^{n} \frac{a_k (\beta(n)^p) \hat{f(k)}}{A_n}|^p \beta(n)^p \leq (F_1 + F_2)^P \leq \epsilon^p.$$

Now by Theorem 2.11 in [10] the weighted mean operator matrix A is compact. So the proof is completed. □

Lemma 2.3. *Let $\{a_k\}, \{b_n\}$ be nonnegative sequences and $p > 1$. Then for all $m_1 \in \mathbb{N}$,*

$$\sum_{n=m_1}^{m} b_n (\sum_{k=n}^{m} a_k)^p \leq p \sum_{k=m_1}^{m} a_k (\sum_{n=m_1}^{k} b_n)(\sum_{j=k}^{m} a_k)^{p-1}.$$

Proof. Lemma 2.2.4 in [3] implies that

$$(\sum_{k=n}^{m} a_k)^p \leq p \sum_{k=n}^{m} a_k (\sum_{j=k}^{m} a_j)^{p-1}.$$

Define $a_{nk} = a_k b_n$ for $m_1 \leq n \leq m, n \leq k \leq m$. By using induction on m, let$\sum_{n=m_1}^{m} \sum_{k=n}^{m} a_{nk} = \sum_{k=m_1}^{m} \sum_{n=m_1}^{k} a_{nk}$ be true and note that

$$\begin{aligned}\sum_{n=m_1}^{m+1} \sum_{k=n}^{m+1} a_{nk} &= \sum_{n=m_1}^{m} \sum_{k=n}^{m} a_{nk} + \sum_{n=0}^{m+1} a_{n(m+1)} \\ &= \sum_{k=m_1}^{m} \sum_{n=m_1}^{k} a_{nk} + \sum_{n=0}^{m+1} a_{n(m+1)} \\ &= \sum_{k=m_1}^{m+1} \sum_{n=m_1}^{k} a_{nk}.\end{aligned}$$

So we get

$$\sum_{n=m_1}^{m} b_n \sum_{k=n}^{m} a_k = \sum_{n=m_1}^{m} \sum_{k=n}^{m} a_k b_n = \sum_{k=m_1}^{m} a_k \sum_{n=m_1}^{k} b_n.$$

Thus

$$\begin{aligned}\sum_{n=m_1}^{m} b_n (\sum_{k=n}^{m} a_k)^p &\leq p \sum_{n=m_1}^{m} b_n \sum_{k=n}^{m} a_k (\sum_{j=k}^{m} a_j)^{p-1} \\ &= p \sum_{k=m_1}^{m} a_k (\sum_{n=m_1}^{k} b_n)(\sum_{j=k}^{m} a_j)^{p-1}.\end{aligned}$$

Now the proof is complete. $\square$

Proposition 2.4. *Let* $\frac{1}{p} + \frac{1}{q} = 1$, $m_1 \in \mathbb{N}$, *and* $\{r_n\}, \{u_n\}, \{v_n\}$ *be nonnegative sequences such that*

$$\lim_{m \longrightarrow \infty} (\sum_{n=m_1}^{m} v_n^{1-q})^{\frac{-1}{p}} (\sum_{n=m_1}^{m} u_n (\sum_{k=m_1}^{\infty} v_k^{1-q})^p)^{\frac{1}{p}} = 0$$

and $(\sum_{n=0}^{\infty} r_n^q u_n^{1-q})^{\frac{1}{q}} < \infty$. *Then*

$$\lim_{m \longrightarrow \infty} (\sum_{k=m}^{\infty} v_k^{1-q} (\sum_{n=k}^{\infty} r_n)^q = 0.$$

Proof. Define

$$G_m = (\sum_{n=m_1}^{m} v_n^{1-q})^{\frac{-1}{p}} (\sum_{n=m_1}^{m} u_n (\sum_{k=m_1}^{\infty} v_k^{1-q})^p)^{\frac{1}{p}}.$$

So $\lim_{m\longrightarrow\infty} G_m = 0$. Without loss of generality assume that $G_m < \infty$, and let $\epsilon > 0$. Then there exists $m_0 \in \mathbb{N}$ such that $G_m < \epsilon$ for all $m \geq m_0$. Put $S = \sum_{k=m}^{\infty}(\sum_{n=k}^{\infty} r_k)^q v_k^{1-q}$. By Lemma 2.3 we get

$$\begin{aligned} S &\leq q \sum_{n=m}^{\infty} r_n (\sum_{k=m}^{n} v_k^{1-q})(\sum_{j=n}^{\infty} r_j)^{q-1} \\ &= q \sum_{n=m}^{\infty} r_n u_n^{\frac{1-q}{q}} (\sum_{k=m}^{n} v_k^{1-q})(\sum_{j=n}^{\infty} r_j)^{q-1} u_n^{\frac{q-1}{q}}. \end{aligned}$$

Now, Hölder's inequality implies that

$$S \leq q(\sum_{n=m}^{\infty} r_n^q u_n^{1-q})^{\frac{1}{q}} (\sum_{n=m}^{\infty} u_n (\sum_{k=m}^{n} v_k^{1-q})^p (\sum_{j=n}^{\infty} r_j)^{p(q-1)})^{\frac{1}{p}}.$$

Put

$$S_1 = \sum_{n=m}^{\infty} u_n (\sum_{k=m}^{n} v_k^{1-q})^p (\sum_{j=n}^{\infty} r_j)^{p(q-1)}.$$

Since $G_m < \epsilon$ for $j \geq n \geq m \geq m_0$, we get

$$\begin{aligned} S_1 &= \sum_{n=m}^{\infty} u_n \sum_{j=n}^{\infty} r_j (\sum_{s=n}^{\infty} r_s)^{p(q-1)-1} (\sum_{k=m}^{n} v_k^{1-q})^p \\ &\leq G_j^p \sum_{j=m}^{\infty} r_j \sum_{n=m}^{j} v_n^{1-q} (\sum_{s=n}^{\infty} r_s)^{p(q-1)-1} \\ &\leq \epsilon^p \sum_{n=m}^{\infty} v_n^{1-q} (\sum_{j=n}^{\infty} r_j)^q. \end{aligned}$$

Thus for $j \geq n \geq m \geq m_0$, we get

$$S \leq q(\sum_{n=m}^{\infty} r_n^q u_n^{1-q})^{\frac{1}{q}} (\epsilon^p (\sum_{n=m}^{\infty} v_n^{1-q} (\sum_{j=n}^{\infty} r_j)^q)^{\frac{1}{p}},$$

and so

$$\sum_{k=m}^{\infty} v_k^{1-q} (\sum_{n=k}^{\infty} r_k)^q \leq (q\epsilon)^q \sum_{n=m}^{\infty} r_n^q u_n^{1-q}.$$

Now the proof is complete. □

Theorem 2.5. *If* $\frac{1}{p}+\frac{1}{q}=1$, $m_1\in\mathbb{N}$, *and*

$$\lim_{m\longrightarrow\infty}(\sum_{n=m_1}^{m}\beta(n)^p a_n^q)^{\frac{-1}{p}}(\sum_{n=m_1}^{m}\beta(n)^p A_n^{\frac{q}{1-q}}(\sum_{k=m_1}^{\infty}\beta(k)^p a_k^q)^p)^{\frac{1}{p}}=0.$$

Then the bounded adjoint of the mean operator matrix A is compact on $H^p(\beta)$.

Proof. Let $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n\in B(H^q(\beta^{\frac{p}{q}}))$, thus

$$A^*(f)(z)=\sum_{k=0}^{\infty}(\sum_{n=k}^{\infty}a_k\frac{\beta(n)^p}{A_n}\hat{f(n)})z^k\in A^*[B(H^q(\beta^{\frac{p}{q}})].$$

In proposition 2.4, put $r_n=\frac{\hat{f}(n)\beta(n)^p}{A_n}$, $u_n=\beta(n)^p A_n^{\frac{q}{1-q}}$, $v_k^{1-p}=\beta(k)^p a_k^q$, and

$$G_m=(\sum_{n=m_1}^{m}v_n^{1-q})^{\frac{-1}{p}}(\sum_{n=m_1}^{m}u_n(\sum_{k=m_1}^{\infty}v_k^{1-q})^p)^{\frac{1}{p}}.$$

Note that $\lim\limits_{m\longrightarrow\infty}G_m=0$. So for every $\epsilon>0$, there exists $m_0\in\mathbb{N}$ such that $G_m<(\frac{\epsilon}{q^qM})^{\frac{1}{q}}$ for all $m_0\leq m$. By [10, Theorem 2.11] we get

$$\begin{aligned}\sum_{k=m}^{\infty}|A^*\hat{(f)}(k)|^q\beta(k)^p &= \sum_{k=m}^{\infty}\beta(k)^p a_k^q(\sum_{n=k}^{\infty}\frac{\beta(n)^p|\hat{f(n)}|}{A_n})^q\\ &= \sum_{k=m}^{\infty}v_k^{1-q}(\sum_{n=k}^{\infty}r_n)^q\leq(q\epsilon)^q\sum_{n=m}^{\infty}r_n^q u_n^{1-q}\\ &\leq(q\epsilon)^q(\sum_{n=m}^{\infty}\beta(n)^p|\hat{f}(n)|^q)\leq(q\epsilon)^q\|f\|^q.\end{aligned}$$

Now the proof is complete. □

References

[1] I. Djolovic, E. Mallkowsky, A note on compact operators on matrix domains, *J. Math. Annal. Appl.*, **340** (2008), 291-303.

[2] E. Mallkowsky, V. Rakocevic, On matrix domains of triangles, *Appl. Math. Comput.*, **189**, No. 2 (2007), 1146-1163.

[3] B.G. Pachpatte, *Mathematical Inequalities*, Elsevier (2005).

[4] W.L.C. Sargent, On compact matrix transformations between sectionally bounded BK-spaces, *Journal London Math. Soc.*, **41** (1996), 79-86.

[5] B. Yousefi, On the space $l^p(\beta)$, *Rendiconti Del Circolo Matematico Di Palermo*, Serie II, Tomo XLIX (2000), 115-120.

[6] B. Yousefi, Unicellularity of the multiplication operator on Banach spaces of formal power series, *Studia Mathematica*, **147**, No. 3 (2001), 293-302.

[7] B. Yousefi, Bounded analytic structure of the Banach spaces of formal power series, *Rend. Circ. Mat. Palermo*, Serie II, Tomo LI (2002), 403-410.

[8] B. Yousefi, S. Jahedi, Composition operators on Banach spaces of formal power series, *Bollettino Della Unione Mthematical Italiana*, **8** (2003), 481-487.

[9] B. Yousefi, Strictly cyclic algebra of operators acting on Banach spaces $H^p(\beta)$, *Czechoslovak Mathematical Journal*, **54**, No. 129 (2004), 261-266.

[10] B. Yousefi, E. Pazouki, Boundedness and compactness of the mean operator matrix on weighted Hardy spaces, *ISRN Mathematical Analysis*, **2012**, Article ID 945741, 13 pages, doi: 10.5402/2012/945741.