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A COMBINED DESCENT GRADIENT METHOD AND DISCRETIZATION METHOD FOR CONVEX SIP

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Abstract: We propose a new method for solving convex semi-infinite problems by using a combination of two methods, the first is the descent gradient method and the second is the discretization method. At each iteration the value of the objective function decreases and we have a feasible solution, for certain problems the feasibility is so important as the optimality (e.g. in control systems design). If we decide to stop our algorithm after a finite number of iterations, we have an optimal solution or an approximate solution which is feasible.

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1. Introduction

We consider the following problem:

$$(SIP) \left\{ \begin{array}{c} \min f(x) \\ g(x,s) \leq 0, \forall s \in S \subset R \\ x \in X \in R^n \end{array} \right.$$

with $X = [x^l, x^u] \subset R^n, x^l < x^u$ (i.e. componentwise), S a compact of R,

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$f \in C^2(R^n, R)$ and $g \in C^2(R^n \times R, R)$, f and g are convex with respect to x. We assume that g is Lipschitzian with respect to x (i.e $\exists L > 0$ such that $|g(x,s) - g(y,s)| \leq L||x-y||, \forall x, y \in X \in R^n, \forall s \in S$) and the Slater condition is satisfied (i.e. $\exists \widetilde{x}$ such that $g(\widetilde{x}, s) < 0, \forall s \in S$). Problems of this type, in which a finite-dimensional decision variables is subject to infinitely many inequality constraints, are called semi-infinite. The existing methods to solve (SIP) problem like the exchange method, the discretization method, the reduction method are based on the reduction to a sequence of constrained nonlinear programming, see the paper [5] which has a fairly complete bibliography on the subject. In fact, discretization and exchange methods approximate the feasible set of (SIP) by finitely many inequalities corresponding to finitely many indices in S, yielding an outer approximation of this set, and reduction-based methods solve the Karush-Kuhn-Tucker system of (SIP) by a Newton-SQP approach. As a consequence, the iterates of these methods are not necessarily feasible for (SIP), but only their limit might be. In [1] the interval method is used for the upper bound and a relaxation method for the discretized problem is used to compute the lower bound. In [4] an adaptive convexification algorithm for a feasible point is presented which consists in the use of the αBB method for the lower problem, the goal of the method is to compute a stationnary method of semi-infinite problem, the index set is equal to $[0, 1]$ while in [11] they consider the problem with an arbitrary index sets. In [9] for the lower bound they use the discretization method with the first and the second necessary conditions for the upper bound, the Mc Cormick relaxation is used for the lower level problem. An hybrid algorithm is used in [2] which combines the deterministic and stochastic global optimization algorithms, a penalty method is applied for the semi-infinite problem, the genetic algorithm solves the penalty problem while the penalty term is computed by the interval analysis method. In [8] a homotopy method is considered which consists in the reformulation of the SIP as two systems of KKT, he ensures the feasibility of the stationnary point of SIP problem by using a homotopy function. Techniques of global optimization are used in [6] for semi-infinite and generalized semi-infinite programs, for the later the index set depends on x. Our method is a combination of the descent gradient method and the discretization method, the iterates generated by our algorithm are always feasible, the objective function value is decreased at each iteration. The paper is organized as follows: In Section 2, we present a method how to construct an upper bound function. Section 3 contains the different steps of our method, the algorithm and its convergence are presented in Section 4. A numerical example found in the literature is treated in detail in Section 5.

2. Upper Bound Function

We now explain how to construct an upper bound function of a function of class c^2 on an interval $[a,b]$. We solve the global optimization problem $\max_{s\in S} g(x^k,s)$ (approximately) by incomplete Branch and Bound which is a classical Branch and Bound to which we add the following stopping rules to check the feasibility:

1) $LB_k > 0$ the solution is not feasible.

2) $UB_k \leq 0$ the solution is feasible.

3) $UB_k < 0$ the solution is strictly feasible.

For $m \geq 2$, let $\{v_1, v_2, ..., v_m\}$ be linear functions (see [3]):

$$v_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} & \text{if } x_{i-1} \leq x \leq x_i \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & \text{if } x_i \leq x \leq x_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

One has

$$\sum_{i=1}^{i=m} v_i(x) = 1, \forall x \in [a,b] \text{ and } v_i(x_j) = 0 \text{ if } i \neq j, \ 1, \text{ otherwise.}$$

Let $L_h f$ be the interpolant function of f at the points $x_1, x_2, ..., x_m$:

$$L_h f(x) = \sum_{i=1}^{i=m} f(x_i) v_i(x).$$

The following result of [3] gives an upper bound and a lower bound of f on an interval $[a,b]$, $(h = b - a)$.

Theorem 2.1. (see [3]) $\forall x \in [a,b]\,, |L_h f(x) - f(x)| \leq \frac{1}{8}Kh^2$, *i.e.*

$$L_h f(x) - \frac{1}{8}Kh^2 \leq f(x) \leq L_h f(x) + \frac{1}{8}Kh^2.$$

In [7] a lower bound function of f is proposed:

$$Lf(x) := L_h f(x) - \frac{1}{2}K(x-a)(b-x) \leq f(x), \forall x \in [a,b]\,.$$

We have shown that this lower bound function is better than that of [3]:

$$Lf(x) \geq L_h f(x) - \frac{1}{8}Kh^2.$$

We now introduce a quadratic upper bound function of f :

Theorem 2.2. $\forall x \in [a,b]$, *we have*

$$L_h f(x) + \frac{1}{8}Kh^2 \geq Uf(x) := L_h f(x) + \frac{1}{2}K(x-a)(b-x) \geq f(x). \tag{2.1}$$

Proof. Let $E(x)$ be the function defined on $[a,b]$ by

$$E(x) = L_h f(x) + \frac{1}{8}Kh^2 - Uf(x).$$

E is convex on $[a,b]$, and its first derivative vanishes at the point $x^* = \frac{1}{2}(a+b)$. $\forall x \in [a,b]$ we have

$$E(x) \geq \min\{E(x) : x \in [a,b]\} = E(x^*) = 0.$$

Then, the first inequality in (1) is satisfied. Consider now the function ϕ defined on S by

$$\phi(x) := Uf(x) - f(x) = L_h(x) + \frac{1}{2}K(x-a)(b-x) - f(x).$$

It is clair that $\phi''(x) = -K - f''(x) \leq 0, \forall x \in S$. So ϕ is concave function, $\forall x \in [a,b]$ we have

$$\phi(x) \geq \min\{\phi(x) : x \in [a,b]\} = \phi(a) = \phi(b) = 0.$$

The second inequality in (2.1) is then proved. □

3. The Method

Step 1: We solve the discretized problem including the end points of the interval S and we check the feasibility by incomplete Branch and Bound applied for the global optimization problem $max_{s\in S} g(x^k, s)$

We have two cases:

First case: The solution is feasible, then it is optimal.

Second case: The solution is not feasible, then the solution of (SIP) problem is on the boundary of the feasible set.

Step 2: We search a strictly feasible point by using the following method. Suppose, we are at the iteration k, we have not yet found a strictly feasible point, and we solve the following $2n$ convex problems $(i = 1, .., n)$:

$$\left(P_{H_k}^{i,\min}\right) \begin{cases} \min x_i \\ g(x, s_j) \leq 0, \;\; j = 1, .., m \\ g(x, \;\; s_{x_{H_t}}) \leq 0, t = 0, ..., k-1 \end{cases},$$

$$\left(P_{H_k}^{i,\max}\right) \begin{cases} \max x_i \\ g(x, s_j) \leq 0, \;\; j = 1, .., m \\ g(x, \;\; s_{x_{H_t}}) \leq 0, t = 0, ..., k-1 \end{cases}.$$

We obtain the hyperrectangle H_k, we compute its center C_{H_k}, and we solve the following problem by incomplete Branch and Bound:

$$\max_{s \in S} g(C_{H_k}, s)$$

If $UB_k < 0$ stop C_{H_k} is strictly feasible.

ii) If $LB_k > 0$, then C_{H_k} is not feasible and we add $s_{C_{H_k}}$ to the next discretization and we repeat the procedure.

We have a sequence of nested hyperrectangles which contain the feasible set of (SIP). This will allow us to find a strictly feasible point.

Step 3: Let $\overline{x}^1$ be strictly feasible point, we compute by IBB w_1 by solving

$$\max_s g(\overline{x}^1, s) \leq w_1 \leq \max_s g(\overline{x}^1, s) + 1.$$

At iteration k, we compute

$$\max_{s \in S} g(\overline{x}^k, s) \leq w_k \leq \max_{s \in S} g(\overline{x}^k, s) + \frac{1}{k},$$

we let

$$x^{k+1} = x^k + \alpha_k d_k.$$

If $\frac{-w_k}{L} \geq \frac{1}{k}$,

$$\alpha_k = \frac{1}{k}$$

and $d_k = -\frac{\nabla f(x^k)}{||\nabla f(x^k)||}$, otherwise

$$\alpha_k = \frac{-w_k}{L},$$

and $d_k = \frac{x^{mk} - x^k}{||x^{mk} - x^k||}$, with x^{mk} being the solution of the following discretized problem:

$$\left(P^{mk}\right) \begin{cases} \min f(x) \\ g(x, s_i) \leq 0, i = 1, ..., k \\ x \in R^n \end{cases} .$$

We assume that the sequence x^{mk} obtained by solving the discretized problem converges to an optimal solution x^* of (SIP) problem, for the conditions stated (see [10]).

Theorem 3.1. *Let x^F be any point on the boundary of the feasible set of (SIP), then $||x^F - \overline{x}^k|| \geq \frac{-w_k}{L}$.*

Proof. If x^F is on the boundary of the feasible set of (SIP), there exists $s^F \in S$ such that $g(x^F, s^F) = 0$. We have

$$|g(\overline{x}^k, s^F) - g(x^F, s^F)| \leq L||\overline{x}^k - x^F||,$$

then

$$L||\overline{x}^k - x^F|| \geq (g(x^F, s^F) - g(\overline{x}^k, s^F) \geq -w_k$$

which implies

$$||x^F - \overline{x}^k|| \geq \frac{-w_k}{L},$$

w_k is such that $\max_{s \in S} g(\overline{x}^k, s) \leq w_k < 0$.

Thus all points $\overline{x}^k, k = 0, 1, 2, 3.....$ found by our method are feasible for (SIP). □

4. Algorithm and its Convergence

4.1. Algorithm

Initialization: Let $\varepsilon > 0, S = [s_0, s_1]$, compute L (by the interval method).

Step 1: Solve the following discretized problem

$$(P_0)\begin{cases} \min f(x) \\ g(x,s_i) \leq 0, i=0,1 \\ x \in R^n \end{cases}$$

to have x^{m1}, else infeasibility.

If x^{m1} is feasible stop, it is optimal admissible, else continue:

Step 2: Find a strictly feasible point $\overline{x}^1$.

Step 3: (Iteration $k = 1, 2, 3, ...$).

1i) Compute w_k such that
$\max_{s\in S} g(\overline{x}^k, s) \leq w_k \leq \max_{s\in S} g(\overline{x}^k, s) + \frac{1}{k}$.

1ii) If $-\frac{w_k}{L} \geq \frac{1}{k}$.

$\overline{x}^{k+1} = \overline{x}^{(k)} - \frac{\nabla f(\overline{x}^k)}{k||\nabla f(\overline{x}^k)||}$.

If $||\overline{x}^{k+1} - \overline{x}^k|| \leq \epsilon$

stop $\overline{x}^{k+1}$ is an $\epsilon-$optimale solution,

else $k := k+1$ and return to 1i).

1iii) If $-\frac{w_k}{L} < \frac{1}{k}$

$\overline{x}^{k+1} = \overline{x}^{(k)} + \frac{-w_k(x^{mk} - \overline{x}^k)}{L||x^{mk} - \overline{x}^k||}$

if $f(\overline{x}^{k+1}) - f(x^{mk}) \leq \epsilon$

stop $\overline{x}^{k+1}$ is an $\epsilon-$optimale

else $k := k+1$ and return to 1i).

4.2. Convergence

Theorem 4.1. *The sequence $\overline{x}^k$ converges to an optimal solution of (SIP) problem.*

Proof. We have two cases:

First case: $\frac{-w_k}{L} \geq \frac{1}{k}$.

Let x^* be an optimal solution of (SIP) problem, $\overline{x}$ be the limit point of the sequence $\overline{x}^k$ (i.e. we suppose that the feasible set of (SIP) is bounded),

$$d_k = \frac{\nabla f(\overline{x}^k)}{||\nabla f(\overline{x}^k)||}.$$

One has

$$\begin{aligned}|\overline{x}^{k+1} - x^*||^2 &= ||\overline{x}^k - \alpha_k d_k - x^*||^2 \\ &= ||\overline{x}^k - x^*||^2 - 2\alpha_k d_k(\overline{x}^k - x^*) + (\alpha_k)^2||d_k||^2 \\ &\leq ||\overline{x}^k - x^*||^2 - 2\alpha_k(f(\overline{x}^k) - f(x^*)) + (\alpha_k)^2.\end{aligned}$$

(convexity inequality).

Applying this inequality recursively, one obtains

$$\begin{aligned}||\overline{x}^{k+1} - x^*||^2 &\leq ||\overline{x}^1 - x^*||^2 - 2\sum_{i=1}^{k} \alpha_i(f(\overline{x}^i) - f(x^*)) + \sum_{i=1}^{k}(\alpha_i)^2 \\ &\Rightarrow 2\sum_{i=1}^{k} \alpha_i(f(\overline{x}^k) - f(x^*)) \leq ||\overline{x}^1 - x^*||^2 + \sum_{i=1}^{k}(\alpha_i)^2 - ||\overline{x}^{k+1} - x^*||^2.\end{aligned}$$

One has

$$\begin{aligned}2\min_i(f(\overline{x}^i) - f(x^*))\sum_{i=1}^{k}\alpha_i &\leq 2\sum_{i=1}^{k}\alpha_i(f(\overline{x}^i) - f(x^*)) \\ &\leq ||\overline{x}^1 - x^*||^2 + \sum_{i=1}^{k}(\alpha_i)^2 - ||\overline{x}^k - x^*||^2 \leq ||\overline{x}^1 - x^*||^2 + \sum_{i=1}^{k}(\alpha_i)^2.\end{aligned}$$

Passing to the limit in both sides, one obtains

$$0 \leq (f(\overline{x}) - f(x^*)) \leq \frac{||\overline{x}^1 - x^*||^2 + \sum_{i=1}^{\infty}(\alpha_i)^2}{2\sum_{i=1}^{\infty}\alpha_i} \to 0 \Rightarrow f(\overline{x}) = f(x^*),$$

then $\overline{x}$ is an optimal solution.

Second Case: $\frac{-w_k}{L} < \frac{1}{k}$

$$\overline{x}^{k+1} = \overline{x}^{(k)} + \frac{-w_k(x^{mk} - \overline{x}^k)}{L||x^{mk} - \overline{x}^k||}. \qquad (*)$$

The solution x^{mk} obtained by solving the discretized problem converges to an optimal solution x^* for (SIP). Let $\overline{x}$ be the limit point of $\overline{x}^k$. By the convexity of f and using $(*)$, one has

$$\begin{aligned}0 \leq & f(\overline{x}^k) - f(x^*) \leq (\overline{x}^k - x^*)\nabla f(\overline{x}^k) \\ = & (\overline{x}^k - x^{mk} + x^{mk} - x^*)\nabla f(\overline{x}^k) \\ = & (\overline{x}^k - x^{mk})\nabla f(\overline{x}^k) + (x^{mk} - x^*)\nabla f(\overline{x}^k) \\ = & (\overline{x}^k - \overline{x}^{k+1})\frac{L||x^{mk} - \overline{x}^k||}{w_k}\nabla f(\overline{x}^k) + (x^{mk} - x^*)\nabla f(\overline{x}^k)\end{aligned}$$

and

$$(\overline{x}^{k+1} - \overline{x}^k)\nabla f(\overline{x}^k) \leq f(\overline{x}^{k+1}) - f(\overline{x}^k) \leq 0$$

$$\Rightarrow (\overline{x}^k - \overline{x}^{k+1})\frac{L||x^{mk} - \overline{x}^k||}{w_k}\nabla f(\overline{x}^k) \leq 0$$

(because $w_k < 0$)

$$\Rightarrow 0 \leq f(\overline{x}^k) - f(x^*) \leq (x^{mk} - x^*)\nabla f(x^k) \leq ||x^{mk} - x^*||||\nabla f(\overline{x}^k)|| \to 0$$

when $k \to \infty$. Then $f(\overline{x}) = f(x^*)$ and $\overline{x}$ is an optimal solution of (SIP). $\square$

Remark 4.1. The main drawback of the purely discretization method applied to problem (SIP) is that the approximate solution is not feasible. On the contrary, the algorithm proposed in this paper guarantees the feasibility of the approximate solution of problem (SIP).

5. Numerical Example

Example (see [1]):

$$\left\{\begin{array}{c}\min x_2^2 - 4x_2 \\ x_1\cos(s) + x_2\sin(s) - 1 \leq 0, \forall s \in [0, \pi] \\ x_1, x_2 \in R\end{array}\right. .$$

We solve this discretized problem by using the end points 0 and π. The solution is $(x_1, 2), -1 \leq x_1 \leq 1$. It is not feasible. We compute a strictly feasible point $\overline{x}^1 = (0, 0)$ and $w_1 = -1$, f(0,0)=0. We compute the Lipschitz constant (for the constraint) $L = 1$, $\nabla f(0, 0) = (0, -4)$ and $||\nabla f(0, 0)|| = 4$.

One has $\overline{x}^2 = \overline{x}^1 - \frac{\nabla f(0,0)}{||\nabla f(0,0)||} = (0,1), f(\overline{x}^2) = f(0,1) = -3$, We solve (by IBB) $max_s g(\overline{x}^2, s) = 0$ which implies $w_2 = 0, s^2 = \frac{\pi}{2}$ (i.e. we are in the second case). We solve the discretized problem with the points $0, \pi, \frac{\pi}{2}$ (i.e. search for a direction obtained by discretization method). We find the solution $x^{m1} = (x_1, 1), -1 \leq x_1 \leq 1; f(x^{m1}) = -3$, $f(\overline{x}^2) - f(x^{m1}) = 0$, then $\overline{x}^2 = (0,1)$ is an optimal solution which is the same found in [1].

6. Conclusion

We have proposed a new method for solving convex semi-infinite optimization problems by combining the descent gradient method and the discretization method. The global optimization techniques are used to maintain the feasibility. The convergence of our algorithm is shown and a numerical example found in the literature is treated for comparison.

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