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WEIGHTED MEAN OPERATOR MATRIX ON SEQUENCE SPACES

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Abstract: In this paper, we study the weighted mean operator matrix on $H^p(\beta)$, and we give conditions under which it is bounded and compact.

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1. Introduction

Let $\beta = \{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$ and $1 < p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$\|f\|^p = \|f\|_{\beta}{}^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

The notation

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$$

shall be used whether or not the series converges for any value of z. These are called formal power series and the set of such series is denoted by $H^p(\beta)$. Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_k$ is a basis such that $\|f_k\| = \beta(k)$. Recall that $H^p(\beta)$ is a reflexive Banach space with norm $\|.\|_\beta$ and the dual

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of $H^p(\beta)$ is $H^q(\beta^{\frac{p}{q}})$ where $\frac{1}{p}+\frac{1}{q}=1$ and $\beta^{\frac{p}{q}}=\{\beta(n)^{\frac{p}{q}}\}$. Let $\{a_n\}$ be a sequence of positive numbers, and define $A_n=\sum_{i=0}^{n}a_i\beta(i)^p$. The weighted mean operator matrix associated with the sequence $\{a_n\}$ is represented by the matrix $A=[a_{nk}]_{n,k}$ and is defined by

$$a_{nk}=\begin{cases}\frac{a_k\beta(n)^p}{A_n} & 0\le k\le n\\ 0 & k>n\end{cases}.$$

Let X and Y be normed linear spaces. Suppose T is a linear operator with domain X and range in Y. We say that T is compact if the image $T(B)$ of closed unit ball $B=\{x:\|x\|\le 1\}\subseteq X$ is relatively compact, that is, $cl[T(B)]$ is compact in Y. For some results on the topic, see [1-7].

2. Main Results

In this section we investigate the boundedness and compactness of weighted mean operator matrix on $H^p(\beta)$.

Lemma 2.1. ([3]) *Let $v_k\ge 0$ for $k=0,1,...,$ and $0<r\le 1$. Thus*

$$r\sum_{k=n}^{m}v_k(\sum_{j=k}^{m}v_j)^{r-1}\le(\sum_{k=n}^{m}v_k)^r$$

for all $m\in\mathbb{N}$. Also, if $v_k\ge 0$ for $k=0,1,...,$ and $r>1$, then

$$r\sum_{k=n}^{m}v_k(\sum_{j=k}^{m}v_j)^{r-1}\ge(\sum_{k=n}^{m}v_k)^r$$

for all $m\in\mathbb{N}$.

Lemma 2.2. *Let $a_n\ge 0$, $b_n\ge 0$ for $n=0,1,...,$ and $r>1$. Then*

$$\sum_{n=0}^{m}(\sum_{k=n}^{m}a_k)^r b_n\le r\sum_{n=0}^{m}a_n(\sum_{j=n}^{m}a_j)^{r-1}(\sum_{k=0}^{n}b_k).$$

Proof. By Lemma 2.1 we have

$$\begin{aligned}\sum_{n=0}^{m}(\sum_{k=n}^{m} a_k)^r b_n &\leq r\sum_{n=0}^{m} b_n(\sum_{k=n}^{m} a_k(\sum_{j=k}^{m} a_j)^{r-1})\\ &= r\sum_{k=0}^{m} a_k(\sum_{n=0}^{k} b_n)(\sum_{j=k}^{m} a_j)^{r-1}\\ &= r\sum_{n=0}^{m} a_n(\sum_{k=0}^{n} b_k)(\sum_{j=n}^{m} a_j)^{r-1}.\end{aligned}$$

This completes the proof. □

Proposition 2.3. *Let $\frac{1}{p}+\frac{1}{q}=1$ and $\{r_n\},\{v_n\},\{u_n\}$ be nonnegative sequences. If*

$$\sup_{n\geq 0}(\sum_{k=0}^{n} v_k^{1-q})^{\frac{-1}{p}}(\sum_{k=0}^{n} u_k(\sum_{j=0}^{k} v_j^{1-q})^p)^{\frac{1}{p}} < \infty,$$

then there exists $c>0$ such that

$$(\sum_{n=0}^{m}(\sum_{k=n}^{m} r_k)^q v_n^{1-q})^{\frac{1}{q}} \leq c(\sum_{n=0}^{m} r_n^q u_n^{1-q})^{\frac{1}{q}}$$

for all $m\geq 0$.

Proof. Let $F=(\sum_{n=0}^{m}(\sum_{k=n}^{m} r_k)^q v_n^{1-q})$, then by Lemma 2.2 we get

$$\begin{aligned}F &\leq q\sum_{n=0}^{m} r_n(\sum_{k=0}^{n} v_k^{1-q})(\sum_{j=n}^{m} v_j)^{q-1}\\ &= q\sum_{n=0}^{m} r_n u_n^{\frac{1-q}{q}}(\sum_{k=0}^{n} v_k^{1-q})(\sum_{j=n}^{m} r_j)^{q-1} u_n^{\frac{q-1}{q}}.\end{aligned}$$

Now, by the Hölder inequality we have

$$F \leq q(\sum_{n=0}^{m} r_n^q u_n^{1-q})^{\frac{1}{q}}(\sum_{n=0}^{m}(\sum_{k=0}^{n} v_k^{1-q})^p(\sum_{j=n}^{m} r_j)^{p(q-1)} u_n)^{\frac{1}{p}}.$$

If $F_1=\sum_{n=0}^{m}(\sum_{k=0}^{n} v_k^{1-q})^p(\sum_{j=n}^{m} r_j)^{p(q-1)} u_n$, then

$$F_1=\sum_{n=0}^{m} u_n(\sum_{j=n}^{m} r_j(\sum_{j=n}^{m} r_j)^{p(q-1)-1}(\sum_{k=0}^{n} v_k^{1-q})^p).$$

Thus

$$F_1 = \sum_{j=0}^{m} r_j (\sum_{n=0}^{j} u_n (\sum_{k=0}^{n} v_k^{1-q})^p)(\sum_{s=n}^{m} r_s)^{p(q-1)-1}.$$

Put

$$E = \sup_{n\geq 0} (\sum_{k=0}^{n} v_k^{1-q})^{\frac{-1}{p}} (\sum_{k=0}^{n} u_k (\sum_{j=0}^{k} v_j^{1-q})^p)^{\frac{1}{p}}.$$

Now by using the hypothesis, we obtain

$$\begin{aligned} F_1 &\leq E^p \sum_{j=0}^{m} r_j \sum_{n=0}^{j} v_n^{1-q} (\sum_{s=n}^{m} r_s)^{p(q-1)-1} \\ &\leq E^p \sum_{n=0}^{m} v_n^{1-q} \sum_{j=n}^{m} r_j (\sum_{j=n}^{m} r_j)^{p(q-1)-1}. \end{aligned}$$

Thus we get

$$F_1 \leq E^p \sum_{n=0}^{m} v_n^{1-q} (\sum_{j=n}^{m} r_j)^{p(q-1)}$$

and so

$$F^{\frac{1}{q}} \leq qE(\sum_{n=0}^{m} r_n^q u_n^{1-q})^{\frac{1}{q}}.$$

Put $c = qE$, so the proof is complete. □

Theorem 2.4. *Let $\frac{1}{p} + \frac{1}{q} = 1$. If*

$$E_0 = \sup_{n\geq 0} (\sum_{k=j}^{n} \beta(k)^p A_k^{\frac{q}{1-q}})(\sum_{s=0}^{k} \beta(s)^p a_s^q)^{p-1} < \infty,$$

then A is bounded on $H^P(\beta)$ and $\|A\| \leq qpE_0$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H^q(\beta^{\frac{p}{q}})$, thus

$$A^*(f)(z) = \sum_{n=0}^{\infty} (\sum_{k=n}^{\infty} a_n \frac{\beta(k)^p}{A_k} f\hat{(}k)) z^n.$$

Note that

$$\|A^*f\|^q \le \sum_{n\ge 0} a_n^q (\sum_{k=n}^{\infty} \frac{\beta(k)^p|\hat{f}(k)|}{A_k})^q \beta(n)^p$$
$$\le \sum_{n\ge 0} a_n^q \beta(n)^p (\sum_{k=n}^{\infty} \frac{\beta(k)^p}{A_k}|\hat{f}(k)|)^q.$$

Put

$$r_n = \frac{\beta(n)^p|\hat{f}(n)|}{A_n}, v_n = (a_n^q\beta(n)^p)^{\frac{1}{q-1}}, u_n = \beta(n)^p A_n^{\frac{q}{1-q}}.$$

We will show that r_n, v_n, u_n hold in the hypothesis of Proposition2.3. We have

$$\sum_{k=0}^{n} u_k (\sum_{j=0}^{k} v_j^{1-q})^p \le p \sum_{k=0}^{n} u_k \sum_{j=0}^{k} v_j^{1-q} (\sum_{s=0}^{j} v_s^{1-q})^{p-1}$$
$$= p\sum_{j=0}^{n} v_j^{1-q} (\sum_{k=j}^{n} u_k)(\sum_{s=0}^{k} v_s)^{p-1}$$
$$= p\sum_{j=0}^{n} \beta(j)^p a_j^q (\sum_{k=j}^{n} \beta(k)^p A_k^{\frac{q}{1-q}})(\sum_{s=0}^{k} \beta(s)^p a_s^q)^{p-1}$$
$$\le pE_0 \sum_{j=0}^{n} \beta(j)^p a_j^q = pE_0 \sum_{j=0}^{n} v_j^{1-q}.$$

Therefore,

$$\sup_{n\ge 0}(\sum_{k=0}^{n} v_k^{1-q})^{\frac{-1}{p}} (\sum_{k=0}^{n} u_k (\sum_{j=0}^{k} v_j^{1-q})^p)^{\frac{1}{p}} \le (E_0 p)^{\frac{1}{p}}$$

for all $n \ge 0$. By Proposition 2.3 we have

$$\sum_{n\ge 0} a_n^q \beta(n)^p (\sum_{k=n}^{\infty} \frac{\beta(k)^p}{A_k}|\hat{f}(k)|)^q = (\sum_{n=0}^{m}(\sum_{k=n}^{m} r_k)^q v_n^{1-q})^{\frac{1}{q}}$$
$$\le q(E_0p)^{\frac{1}{p}} (\sum_{n=0}^{\infty} r_n^q u_n^{1-q})^{\frac{1}{q}}$$
$$= q(E_0p)^{\frac{1}{p}} (\sum_{n=0}^{\infty} |\hat{f}(n)|^q \beta(n)^p)^{\frac{1}{q}}.$$

Now one can see that A is bounded on $H^P(\beta)$and $\|A\| \le q(E_0p)^{\frac{1}{p}}$. □

Remark 2.5. Let $c_{ij} \geq 0$ for $0 \leq i \leq m$, $0 \leq j \leq n$, and $p > 1$, then

$$[\sum_{i=0}^{n}(\sum_{j=0}^{m} c_{ij})^p]^{\frac{1}{p}} \leq \sum_{j=0}^{m}(\sum_{i=0}^{n} c_{ij}^p)^{\frac{1}{p}}.$$

Lemma 2.6. ([1]) *Let $r > s \geq 1$ and let $\{u_n\}, \{v_n\}, \{w_n\}$ be nonnegative sequences. If for all $m \geq 0$*

$$\sum_{n=1}^{m} u_n(\sum_{k=1}^{n} v_k)^r \leq (\sum_{k=0}^{n} v_k)^s,$$

then

$$\sum_{n=1}^{\infty} u_n(\sum_{k=1}^{n} v_k w_k)^r \leq (\frac{r}{r-s})^r(\sum_{k=0}^{\infty} v_k w_k^{\frac{r}{s}})^s.$$

Theorem 2.7. *Let $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that λ is a positive number which $p + q\lambda < p^2$ and*

$$\sum_{j=0}^{n}(\frac{a_j}{\beta(j)})^q(\sum_{k=j}^{n} \frac{\beta(k)^{p^2+p}}{A_k^p})^{\frac{q}{p}} \leq (\sum_{j=0}^{n}(\frac{a_j}{\beta(j)})^q)^{\lambda},$$

then A is bounded on $H^P(\beta)$ and $\|A\| \leq (\frac{p}{p-(1+\frac{\lambda q}{p})})$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \in H_p(\beta)$, thus

$$A(f)(z) = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} \frac{a_k\beta(n)^p}{A_n} f\hat{(k)})z^n.$$

Put

$$u_n = \frac{\beta(n)^{p^2+p}}{A_n^p}, v_n = \frac{a_k}{\beta(k)})^q, w_n = a_n^{\frac{1}{1-p}}|\hat{f}(n)|\beta(n)^q, r = p, s = 1 + \frac{\lambda q}{p}.$$

By the general Minkowski inequality and the Hölder inequality, we will show that u_n, v_n, w_n, r, s hold in the hypothesis of Lemma 2.6. We have

$$\begin{aligned}
\sum_{n=1}^{m} u_n(\sum_{k=1}^{n} v_k)^r &= \sum_{n=0}^{m} \frac{\beta(n)^{p^2+p}}{A_n^p}(\sum_{k=0}^{n}(\frac{a_k}{\beta(k)})^q)^p \\
&\leq (\sum_{k=0}^{m}(\frac{a_k}{\beta(k)})^q(\sum_{n=k}^{m}\frac{\beta(n)^{p^2+p}}{A_n^p})^{\frac{1}{p}})^p \\
&\leq \sum_{k=0}^{m}(\frac{a_k}{\beta(k)})^q(\sum_{k=0}^{m}(\frac{a_k}{\beta(k)})^q(\sum_{n=k}^{m}\frac{\beta(n)^{p^2+p}}{A_n^p})^{\frac{q}{p}})^{\frac{p}{q}} \\
&\leq (\sum_{k=0}^{m}(\frac{a_k}{\beta(k)})^q)^{1+\frac{\lambda q}{p}} = (\sum_{k=0}^{n} v_k)^s.
\end{aligned}$$

Now by using Lemma 2.6 we obtain

$$\begin{aligned}
\|Af\|^p &\leq \sum_{n\geq 0}(\sum_{k=0}^{n}\frac{a_k\beta(n)^p|\hat{f}(k)|}{A_n{}^p})\beta(n)^p \\
&= \sum_{n=1}^{\infty} u_n(\sum_{k=1}^{n} v_k w_k)^r \\
&\leq (\frac{r}{r-s})^r(\sum_{k=0}^{\infty} v_k w_k^{\frac{r}{s}})^s \\
&\leq (\frac{p}{p-(1+\frac{\lambda q}{p})})^p(\sum_{k=0}^{\infty}(a_k^{\frac{1}{1-p}}|\hat{f}(k)|\beta(k)^q)^p a_k^q\beta(k)^{-q})^p.
\end{aligned}$$

Thus A is bounded on $H^p(\beta)$ and $\|A\| \leq (\frac{p}{p-(1+\frac{\lambda q}{p})})$. So the proof is complete. □

Theorem 2.8. *If* $\frac{1}{p}+\frac{1}{q}=1$, *and*

$$\lim_{m\longrightarrow\infty}\sum_{k=m}^{\infty}(\sum_{n=k}^{\infty}(\frac{\beta(n)^{p^2+p}a_k^p}{A_n^p\beta(k)^p})^q = 0.$$

Then the bounded mean operator matrix A *is compact on* $H^p(\beta)$.

Proof. Without loss of generality assume that $f \geq 0$ and $\|f\|_\beta \leq 1$. Define the matrix A_m with entries e_{nk} where $e_{nk} = 0$, if $0 \leq n < m, 0 \leq k < m$, and

$e_{nk} = a_{nk}$, otherwise. Let $\epsilon > 0$, then there exists $m_0 \in \mathbb{N}$ such that

$$\sum_{k=m}^{\infty}(\sum_{n=k}^{\infty}\frac{\beta(n)^{p^2+p}a_k^q}{A_n^p\beta(k)^p})^q < \epsilon^q,$$

for all $m \geq m_0$. Note that

$$\begin{aligned}\|A_m(f)\|^p &= \sum_{n=m}^{\infty}(\sum_{k=m}^{n}\frac{a_k\beta(n)^{p+1}}{A_n}\hat{f}(k)(\sum_{k=m}^{n}\frac{a_k\beta(n)^p}{A_n}\hat{f}(k))^{p-1}\beta(n)^{p-1}\\ &\leq \sum_{k=m}^{\infty}\hat{f}(k)\beta(k)\|A_m(f)\|^{\frac{p}{q}}(\sum_{n=k}^{\infty}(\frac{a_k\beta(n)^{p+1}}{A_n\beta(k)})^p)^{\frac{1}{p}}\\ &\leq \|A_m(f)\|^{\frac{p}{q}}\|f\|\epsilon.\end{aligned}$$

Thus $\|A_m\| < \epsilon$ for all $m \geq m_0$, and $\|f\| \leq 1$. So $\lim_{m\longrightarrow\infty} A_m = 0$. Now, consider the matrix $B_m = [b_{nk}]$ where $b_{nk} = a_{nk}$, if $0 \leq n, k < m$, and $b_{nk} = 0$, otherwise. Thus for all m, B_m is a finite rank operator on $H^p(\beta)$, and we have $\lim_{m\longrightarrow\infty} \|A_m\| = \|B_m - A\| = 0$. Thus A is a compact operator on $H^p(\beta)$. □

Example 2.9. If $a_n = \beta(n) = 1$ and $p > 1$, then $A_n = n + 1$. So

$$\sum_{n=k}^{\infty}\frac{\beta(n)^{p^2+p}a_k^p}{A_n^p\beta(k)^p} = \sum_{n=k}^{\infty}\frac{1}{(n+1)^p} = \frac{1}{(p-1)(k+1)^{p-1}},$$

and

$$\sum_{k=m}^{\infty}(\sum_{n=k}^{\infty}\frac{\beta(n)^{p^2+p}a_k^q}{A_n^p\beta(k)^p})^q = \sum_{k=m}^{\infty}\frac{1}{((p-1)(k+1)^{p-1})^q}.$$

Thus by Theorem 2.8, A is a compact operator.

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