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FOURIER SERIES FOR QUATERNIONS AND THE SQUARE OF THE ERROR THEOREM

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Abstract: In this paper we introduce a type of Hypercomplex Fourier Series based on Quaternions, and discuss on a Hypercomplex version of the Square of the Error Theorem. Since their discovery by Hamilton (Sinegre [1]), quaternions have provided beautifully insights either on the structure of different areas of Mathematics or in the connections of Mathematics with other fields. For instance: I) Pauli spin matrices used in Physics can be easily explained through quaternions analysis (Lan [2]); II) Fundamental theorem of Algebra (Eilenberg [3]), which asserts that the polynomial analysis in quaternions maps into itself the four dimensional sphere of all real quaternions, with the point infinity added, and the degree of this map is n. Motivated on earlier works by two of us

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on Power Series (Pendeza et al. [4]), and in a recent paper on Liouville's Theorem (Borges and Marõ [5]), we obtain an Hypercomplex version of the Fourier Series, which hopefully can be used for the treatment of hypergeometric partial differential equations such as the dumped harmonic oscillation.

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1. Introduction and Motivation

Physical problems as the conduction patterns of heat and the study of vibrations and oscillations led to the study of the classical Fourier series. The Fourier series may be regarded as an infinite sum of trigonometric functions that can be used to model real-valued, periodical functions. Specifically, it can break up a periodic function into an infinite series of sines and cosines waves. One of the useful properties of the oscillating systems that makes the Fourier series useful is the property of superposition - in other words, suppose the driving force $f(t)$ along with some initial conditions, produces some steady solution $x(t)$, and another driving force, $g(t)$ produces the steady solution $y(t)$. Then, the driving force $h(t) = f(t) + g(t)$ produces the steady-state response $z(t) = x(t) + y(t)$.

The Fourier series are of great importance in theoretical mathematics too. Of the many possible methods of estimating complex-valued functions, Fourier series are specifically attractive because uniform convergence of the Fourier Series (as more terms are added) is guaranteed for continuous, bounded functions. Furthermore, the Fourier coefficients are designed to minimize the square of the error from the actual function. Finally complex exponentials are relatively simple to deal with and ubiquitous in physical phenomena.

Nowadays, the Fourier series have been studied in the context of Hypercomplex analysis. For instance, Bock and Gurlebeck [6], have described recently an Appell system that is orthonormal and complete in the space of square integrable quaternion-valued functions in the unit ball of R^3. The authors presented a new Taylor-type Series expansion, based on the Appell polynomials, which can be related to the corresponding Fourier series.The results proved in their paper are very useful since in the last few decades the theory of monogenic functions has found several applications to boundary value problems, extended recently to the case of initial-boundary value problems.

In our paper, based on a previous work by two of us (Pendeza and Borges [4]), a simple quaternionic expansion of Fourier Series in hypercomplex exponentials is provided. We show in some examples of how to deal with our expansion, with the purpose of treating with practical problems in the nearest future. It is also obtained a hypercomplex version of a well known Walter Rudin's (Rudin [7]) complex theorem for the square of the error of Fourier series.

2. Hypercomplex Fourier Series

In this section we present the hypercomplex Fourier series. This presentation will be made using the classic formulation of Fourier series and the quaternionic base. In this sense, we begin this section by considering f a function defined on interval $[-L, L]$, $L > 0$, and outside of this interval set as $f(x) = f(x+2L)$, that is, $f(x)$ is $2L-$periodical. If f and f' are piecewise continuous, then the series of function given below,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \tag{1}$$

is convergent and the limit is $\widetilde{f}(x) = \dfrac{\lim_{a\to x^+} f(x) + \lim_{a\to x^-} f(x)}{2}$. The coefficients a_0, a_n and b_n are the Fourier coefficients of f given by:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx, \tag{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \tag{3}$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{4}$$

The trigonometric series presented in (1) with this choice of coefficients is the Fourier series of f.

Now, let us present some properties of the exponential function which will be useful. First, note that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \ldots \tag{5}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \tag{6}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}. \tag{7}$$

Using the definition of e^z with $z \in \mathbb{C}$ we have

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \dots & (8)\\ &= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} - \frac{y^6}{6!} - \dots & (9)\\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \dots\right) & (10)\end{aligned}$$

and from (6) and (7), we obtain

$$e^{iy} = \cos y + i \sin y.$$

Thus, given the property of the exponential $x^a.x^b = x^{a+b}$, we can define the complex exponential e^z as follows

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

Now, we will detail some properties considering a quaternion given by $q = x + y_1 i + y_2 j + y_3 k$. We begin these properties by pointing out that $\forall q \in \mathbb{C}$ we have $q \in \mathbb{Q}$. In fact, we just consider $y_2 = 0 = y_3$ and the implication is immediate.

Writing

$$e^q = e^{x+y_1 i+y_2 j+y_3 k} = e^x e^{y_1 i} e^{y_2 j} e^{y_3 k}, \tag{11}$$

we have

$$e^q = e^x[(\cos y_1 + i \sin y_1)(\cos y_2 + j \sin y_2)(\cos y_3 + k \sin y_3)]. \tag{12}$$

Since $\cos y = \frac{e^{iy}+e^{-iy}}{2}$ and $\sin y = \frac{e^{iy}-e^{-iy}}{2}$ we can get

$$\begin{aligned} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) &= a_n\left[\frac{e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}}{2}\right] + b_n\left[\frac{e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}}{2}\right] \\ &= \underbrace{\left(\frac{a_n}{2} + \frac{b_n}{2i}\right)}_{c_n^1} e^{\frac{in\pi x}{L}} + \underbrace{\left(\frac{a_n}{2} - \frac{b_n}{2i}\right)}_{c_{-n}^1} e^{\frac{-in\pi x}{L}}. \end{aligned}$$

Note that

$$c_n^1 = \frac{a_n}{2} + \frac{b_n}{2i} = \frac{1}{2}(a_n - ib_n).$$

Then, using (3) and (4) we obtain

$$c_n^1 = \frac{1}{2}(a_n - ib_n) = \frac{1}{2}\left[\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx - i\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right]$$
$$= \frac{1}{2L}\int_{-L}^{L} fa(x)\left[\cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)\right]dx.$$

Thus

$$c_n^1 = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i(\frac{n\pi x}{L})}dx. \tag{13}$$

Using the same argument we can get

$$\begin{aligned} c_{-n}^1 = \frac{a_n}{2} - \frac{b_n}{2i} &= \frac{1}{2}(a_n + ib_n) \\ &= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\cos\left(\frac{n\pi x}{L}\right) + i\sin\left(\frac{n\pi x}{L}\right)\right]dx \end{aligned}$$

and consequently,

$$c_{-n}^1 = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\left(\frac{(-n)\pi x}{L}\right)}dx. \tag{14}$$

From (13) and (14), we can rewrite the series presented in (1) as follows

$$\begin{aligned} \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right] &= \sum_{n=1}^{\infty}\left[c_n^1 e^{i\frac{n\pi x}{L}} + c_{-n}^1 e^{-i\frac{n\pi x}{L}}\right] \\ &= \sum_{n=1}^{\infty} c_n^1 e^{i\frac{n\pi x}{L}} + \sum_{n=1}^{\infty} c_{-n}^1 e^{\frac{-in\pi x}{L}} \\ &= \sum_{-\infty}^{-1} c_n^1 e^{i\frac{n\pi x}{L}} + \sum_{n=1}^{\infty} c_n^1 e^{\frac{in\pi x}{L}} \\ &= \sum_{-\infty, n\neq 0}^{\infty} c_n^1 e^{\frac{in\pi x}{L}}. \end{aligned}$$

Considering

$$c_0 = \frac{a_0}{2},$$

we have

$$\widetilde{f}(x) = c_o + \sum_{-\infty, n\neq 0}^{\infty} c_n^1 e^{\frac{in\pi x}{L}}, \tag{15}$$

that is, we express f as a series of complex functions.

Considering the base quaternionic $\{1,\ i,\ j,\ k\}$ and using similar arguments to (13) and (14), we can obtain:

$$c_n^2 = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-j\left(\frac{n\pi x}{L}\right)}dx, \tag{16}$$

and

$$c_{-n}^2 = \frac{1}{2L}\int_{-L}^{L} f(x)e^{j\left(\frac{n\pi x}{L}\right)}dx, \tag{17}$$

since $(j)^2 = -1$.

$$c_n^3 = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-k\left(\frac{n\pi x}{L}\right)}dx, \tag{18}$$

and

$$c_{-n}^3 = \frac{1}{2L}\int_{-L}^{L} f(x)e^{k\left(\frac{n\pi x}{L}\right)}dx, \tag{19}$$

since $(k)^2 = -1$.

Still, from (1) and considering the same argument used to obtain (15), we have

$$\widetilde{f}(x) = c_o + \sum_{-\infty, n\neq 0}^{\infty} c_n^2 e^{\frac{jn\pi x}{L}} \tag{20}$$

and

$$\widetilde{f}(x) = c_o + \sum_{-\infty, n\neq 0}^{\infty} c_n^3 e^{\frac{in\pi x}{L}}. \tag{21}$$

Therefore, we can write $f(x)$ as a sum of (15), (20) and (21)

$$\widetilde{f}(x) = c_o + \frac{1}{3}\left\{ \sum_{-\infty, n\neq 0}^{\infty} c_n^1 e^{\frac{in\pi x}{L}} + \sum_{-\infty, n\neq 0}^{\infty} c_n^2 e^{\frac{jn\pi x}{L}} + \sum_{-\infty, n\neq 0}^{\infty} c_n^3 e^{\frac{kn\pi x}{L}} \right\}. \tag{22}$$

Since $f : \mathbb{R} \longrightarrow \mathbb{R}$ is periodic, with period $2L$, f and f' are piecewise continuous, we have that the Fourier series of f can be written as

$$\begin{aligned}\widetilde{f}(x) = c_o \quad &+ \quad \frac{1}{3}\left\{\sum_{-\infty,n\neq 0}^{\infty} c_n^1 \left[\cos\left(\frac{n\pi x}{L}\right) + i\sin\left(\frac{n\pi x}{L}\right)\right]\right. \\ &+ \quad \sum_{-\infty,n\neq 0}^{\infty} c_n^2 \left[\cos\left(\frac{n\pi x}{L}\right) + j\sin\left(\frac{n\pi x}{L}\right)\right] \\ &+ \quad \left.\sum_{-\infty,n\neq 0}^{\infty} c_n^3 \left[\cos\left(\frac{n\pi x}{L}\right) + k\sin\left(\frac{n\pi x}{L}\right)\right]\right\},\end{aligned}$$

or, in an equivalent way,

$$\widetilde{f}(x) = c_o + \frac{1}{3}\left\{\sum_{-\infty,n\neq 0}^{\infty} c_n^1 e^{\frac{in\pi x}{L}} + c_n^2 e^{\frac{jn\pi x}{L}} + c_n^3 e^{\frac{kn\pi x}{L}}\right\}. \tag{23}$$

The series (23) is the Hypercomplex Fourier series of f.

Example 1. Let

$$f(x) = \begin{cases} 2, & \text{if } 0 \leq x < \pi \\ 0, & \text{if } -\pi < x \leq 0 \\ f(x+2\pi) = 0. & \end{cases} .$$

We will determine the Hypercomplex Fourier series of f. In fact, calculating the coefficients c_0, c_n^1, c_n^2, c_n^3 we have

$$\begin{aligned} c_0 &= \frac{a_0}{2} = \frac{1}{2L}\int_L^{-L} f(x)dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0dx + \int_0^{\pi} 2dx\right] = 1, \\ c_n^1 &= \frac{1}{2\pi}\int_0^{\pi} 2e^{-inx}dx = \frac{i}{n\pi}[\cos(n\pi) - 1] = \frac{i}{n\pi}[(-1)^n - 1], \\ c_n^2 &= \frac{1}{2\pi}\int_0^{\pi} 2e^{-jnx}dx = \frac{j}{n\pi}[\cos(n\pi) - 1] = \frac{j}{n\pi}[(-1)^n - 1], \\ c_n^3 &= \frac{1}{2\pi}\int_0^{\pi} 2e^{-knx}dx = \frac{k}{n\pi}[\cos(n\pi) - 1] = \frac{k}{n\pi}[(-1)^n - 1], \end{aligned}$$

thus

$$\widetilde{f}(x) = 1 + \sum_{-\infty,n\neq 0}^{\infty}\left[\frac{i[(-1)^n - 1]}{n\pi}e^{inx} + \frac{j[(-1)^n - 1]}{n\pi}e^{jnx} + \frac{k[(-1)^n - 1]}{n\pi}e^{knx}\right].$$

Example 2. Let $f(t) = t$, with $t \in (-1, 1)$ and $f(t+2) = f(t)$. The quaternionic coefficients are given by:

$$\begin{aligned}
c_0 &= 0, \\
c_n^1 &= \frac{1}{2}\int_{-1}^{1} te^{-in\pi t}dt = \frac{(-1)^{n+1}}{(in\pi)}, \\
c_n^2 &= \frac{1}{2}\int_{-1}^{1} te^{-jn\pi t}dt = \frac{(-1)^{n+1}}{(jn\pi)}, \\
c_n^3 &= \frac{1}{2}\int_{-1}^{1} te^{-kn\pi t}dt = \frac{(-1)^{n+1}}{(kn\pi)}.
\end{aligned}$$

Then, the Hypercomplex Fourier series of f is

$$\widetilde{f}(t) = \sum_{-\infty, n\neq 0}^{\infty} [\frac{(-1)^{n+1}}{in\pi}e^{in\pi t} + \frac{(-1)^{n+1}}{jn\pi}e^{jn\pi t} + \frac{(-1)^{n+1}}{kn\pi}e^{kn\pi t}].$$

3. Square of the Error Theorem for Hypercomplex Fourier Series

In this section we will present an important property of the hypercomplex Fourier Series, derived from the proof of the complex case of Walter Rudin (see Rudin [7]), the square of the error theorem.

For the purposes of this section, we denote by ϕ the hypercomplex function defined by

$$\phi : \mathbb{R} \to \mathcal{Q},\ \phi(x) = \phi_1(x) + \phi_2(x)i + \phi_3(x)j + \phi_4(x)k,$$

where $\phi_1,\ \phi_2,\ \phi_3$ and ϕ_4 are real functions.

Thus we can consider

$$\overline{\phi(x)} = \phi_1(x) - (\phi_2(x)i + \phi_3(x)j + \phi_4(x)k),$$

$$|\phi_n(x)|^2 = \phi_n(x)\overline{\phi_n(x)} = (\phi_1(x))^2 + (\phi_2(x))^2 + (\phi_3(x))^2 + (\phi_4(x))^2.$$

Definition 3. Let $\{\phi_n\}$, $n = 1, 2, 3, 4, \ldots$ be a sequence of hypercomplex functions on $[a, b]$, such that

$$\int_a^b \phi_n(x)\overline{\phi_m(x)}dx = 0, \quad (n \neq m). \tag{24}$$

Then, we say that $\{\phi_n\}$ is an orthogonal system of hypercomplex functions on $[a, b]$. In addition, if

$$\int_a^b |\phi_n(x)|^2 dx = 1, \tag{25}$$

for all n, we say that $\{\phi_n\}$ is orthonormal. Here are some examples.

Example 4. The sequence of functions

$$\begin{aligned}
\phi_n^1(x) &= \frac{1}{\sqrt{2L}} e^{\frac{in\pi x}{L}}, \ n = 1, 2, 3, \ldots \\
\phi_n^2(x) &= \frac{1}{\sqrt{2L}} e^{\frac{jn\pi x}{L}}, \ n = 1, 2, 3, \ldots \\
\phi_n^3(x) &= \frac{1}{\sqrt{2L}} e^{\frac{kn\pi x}{L}}, \ n = 1, 2, 3, \ldots
\end{aligned}$$

form orthonormal system on $[-L, L]$.

Example 5. The sequence of functions

$$\phi_n(x) = \frac{1}{\sqrt{12L}} \left(e^{\frac{in\pi x}{L}} + e^{\frac{jn\pi x}{L}} + e^{\frac{kn\pi x}{L}} \right), \ n = 1, 2, 3, \ldots$$

form orthonormal system on $[-L, L]$.

Motivated by Example 5, we define

$$c_n^1 + c_n^2 i + c_n^3 j + c_n^4 k = \int_a^b f(t)\overline{\phi_n(t)} dt, \text{ for all } n = 1, 2, 3, \ldots, \tag{26}$$

where $\{\phi_n\}$ is an orthonormal sequence in $[a, b]$ and c_n^1, c_n^2, c_n^3 and c_n^4 are real sequences.

We call c_n the n-th hypercomplex Fourier coefficient of f (relative to $\{\phi_n\}$). We write

$$\sum_1^\infty (c_n^1 + c_n^2 i + c_n^3 j + c_n^4 k)\phi_n(x) \tag{27}$$

and call this series the hypercomplex Fourier series of f.

The following theorem extends (Rudin [7]) results for the case of hypercomplex Fourier series. More specifically we show that the partial sums of hypercomplex Fourier series of f have a certain minimum property. Let us assume that f is a real function.

Theorem 6. *Let $\{\phi_n\}$ be orthonormal sequence of hypercomplex functions on $[a,b]$. Consider the n-th partial sum of the hypercomplex Fourier series of f*

$$s_n(x) = \sum_{m=1}^{n} (c_m^1 + c_m^2 i + c_m^3 j + c_m^4 k)\phi_m(x), \tag{28}$$

where c_m^1, c_m^2, c_m^3 and c_m^4 are given in (26). In addition, define

$$t_n(x) = \sum_{m=1}^{n} (d_m^1 + d_m^2 i + d_m^3 j + d_m^4 k)\phi_m(x), \tag{29}$$

where d_m^1, d_m^2, d_m^3 and d_m^4 are real sequences. Then

$$\int_a^b |f(x) - s_n(x)|^2 dx \leq \int_a^b |f(x) - t_n(x)|^2 dx, \tag{30}$$

and equality holds if and only if

$$d_m^1 = c_m^1,\ d_m^2 = c_m^2,\ d_m^3 = c_m^3 \text{ and } d_m^4 = c_m^4 \quad (m = 1, 2, \ldots, n). \tag{31}$$

That means that among all functions t_n, s_n gives the best possible mean square approximation to f.

Proof. To simplify the notation, let $\int$ be the integral over $[a,b]$ and $\sum$ the sum from 1 to n. From the definition of (29) and (26), we have

$$\begin{aligned} \int f\overline{t_n} &= \int f\left[\overline{\sum (d_m^1 + d_m^2 i + d_m^3 j + d_m^4 k)\phi_m}\right] \\ &= \sum \overline{(d_m^1 + d_m^2 i + d_m^3 j + d_m^4 k)} \int f\overline{\phi_m} \\ &= \sum \overline{(d_m^1 + d_m^2 i + d_m^3 j + d_m^4 k)}(c_m^1 + c_m^2 i + c_m^3 j + c_m^4 k). \end{aligned}$$

Consequently,

$$\int f\overline{t_n} + \int \overline{f} t_n = \sum [2(d_m^1 c_m^1) + 2(d_m^2 c_m^2) + 2(d_m^3 c_m^3) + 2(d_m^4 c_m^4)].$$

Furthermore,

$$\begin{aligned}\int |t_n|^2 &= \int t_n\overline{t_n} \\ &= \int \sum_{m=1}^{n}[(d_m^1 + d_m^2 i + d_m^3 j + d_m^4 k)\phi_m]\overline{\sum_{p=1}^{n}[(d_p^1 + d_p^2 i + d_p^3 j + d_p^4 k)\phi_p]} \\ &= \sum_{m=1}^{n}[(d_m^1)^2 + (d_m^2)^2 + (d_m^3)^2 + (d_m^4)^2],\end{aligned}$$

since $\{\phi_n\}$ is orthonormal. Therefore,

$$\begin{aligned}\int |f - t_n|^2 &= \int |f|^2 - \int f\overline{t_n} - \int \overline{f}t_n + \int |t_n|^2 \\ &= \int |f|^2 - \sum[2(d_m^1 c_m^1) + 2(d_m^2 c_m^2) + 2(d_m^3 c_m^3) + 2(d_m^4 c_m^4)] + \\ &\qquad + \sum_{m=1}^{n}[(d_m^1)^2 + (d_m^2)^2 + (d_m^3)^2 + (d_m^4)^2] \\ &= \int |f|^2 + \sum[(d_m^1 - c_m^1)^2 + (d_m^2 - c_m^2)^2 + (d_m^3 - c_m^3)^2 \\ &\qquad +(d_m^4 - c_m^4)^2] \\ &\qquad - \sum_{m=1}^{n}[(c_m^1)^2 + (c_m^2)^2 + (c_m^3)^2 + (c_m^4)^2]\end{aligned}$$

which is minimized if and only if

$$d_m^1 = c_m^1,\ d_m^2 = c_m^2,\ d_m^3 = c_m^3 \text{ and } d_m^4 = c_m^4 \quad (m = 1, 2, \dots, n).$$

□

4. Concluding Remarks

In this paper we have discussed on a version of Fourier series of quaternionic type, and presented the Square of the Error Theorem. Fourier Series are remarkable in order to approximate functions by a series of ortonormal functions, which could be used for treating and solving partial differential equations as those related, for instance, to the heat conduction problem and the vibrations and oscillations problems. With this motivation, and with the purpose of that many of the theoretical physics representation models are hypergeometric in the context of extra complex and hypercomplex dimensions, we develop a simple version of a hypercomplex Fourier series, which could be, hopefully, be used in a nearest future in the physical higher-dimensional theories.

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