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THE CONJUGACY CLASS OF SYMMETRIC GROUPS

Sadoon Abdullah I.O.[1], Bilal N. Al-Hasanat[2 §]
[1,2]Department of Mathematics
Al Hussein Bin Talal University
Maan, JORDAN
e-mail: sadoon58@yahoo.com
e-mail: bilal_hasanat@yahoo.com

Abstract: The aim of this work is to classify the iteration of counting the conjugacy classes of the symmetric groups S_n. Our classification is illustrated by stating the main Theorem 1.

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1. Introduction

Symmetric group representation theory is a case of the representation theory of groups. In particular, symmetric groups allow access to specific and detailed theory, such as Galois theory, invariant theory, the representation theory of Lie groups.

Edith and Verrill [4], proved that a product conjugacy class of S_n is never a conjugacy class, and if n is odd and a multiple of three, then the product will be the union of at least three distinct conjugate classes. Formal accounts of group theory and in-depth analysis of its applications in the arts and particularly in the visual arts and architecture have been given in various sources, and several

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§Correspondence author

of them are mentioned in [2].

2. Definitions and Notations

Let G be a finite group with identity element e, for x and y in G, we say that x is conjugate to y if there exists $g \in G$ such that $y = gxg^{-1}$. Let $\sim$ denote the conjugate relation on G, then $(G, \sim)$ is an equivalent relation. That is, x is conjugate to itself for all x in G, so $x \sim x \;\; , \forall x \in G$. If $x \sim y$ then there exists $g \in G$ such that $y = gxg^{-1}$, then $g^{-1}yg = x$, therefore $y \sim x$. If $x \sim y$ and $y \sim z$, then there exist $g, h \in G$ such that $y = gxg^{-1} \;,\;\; z = hyh^{-1}$, then $z = hgxg^{-1}h^{-1} = (hg)x(hg)^{-1}$, it follows $x \sim z$.

Remark 1. Let $x, y \in G$, if $x \sim y$, then x and y have the same order (i.e. if $x \sim y$ and $x^n = e$, then $y^n = e$).

Proof. Let $x, y \in G$ be such that $x \sim y$, then there exists $g \in G$ such that $y = gxg^{-1}$. Let x have a finite order n, then $y^n = \left(gxg^{-1}\right)^n = g^n x^n (g^{-1})^n = g^n e (g^n)^{-1} = e$, then $y^n = e$. To show that n is the smallest such integer, suppose in the contrary that $n > k$ is a positive integer and $y^k = e$, then $e = \left(gxg^{-1}\right)^k = g^k x^k (g^{-1})^k$, solving this equation in terms of x, we get $x \sim e$, thus $x = e$, which a contradiction. So n is the smallest positive integer such that $y^n = e$. □

3. The Conjugacy Classes of S_n

Let S_n be a symmetric group of n sides, then S_n contains $n!$ cycles with identity cycle $() = e$.

The conjugacy class of S_n can be written as follows:

$$cl_{S_n} = \bigcup_{c \in S_n} cl(c).$$

The elements of S_n are called cycles and a cycle is a permutation of m elements, where $0 \leq m \leq n$ and $m \neq 1$.

Let us denote the class of all m-cycles by l_m, where $|l_m| = \dfrac{n!}{(n-m)!}$. Then if we collect all l_m, we get the following set

$$L = \{l_m \mid 0 \leq m \leq n \quad \text{and } m \neq 1\} = \{0,\ 2,\ 3,\ 4, \cdots, n\}.$$

Our aim is to get and count the conjugacy classes for the symmetric groups, and to do this we have first going through the following remarks.

Remark 2. All the m-cycles determine the same conjugacy class.

From another point of view, all cycles of the same length coincide width the same conjugacy class. So no need to find the conjugacy classes for all elements of S_n, we need to find the conjugacy classes just for each family in L.

Remark 3. If X and Y in L are two distinct classes, then $cl(X) \bigcap cl(Y) = \emptyset$.

Proof. Let $X, Y \in L$, then $cl(X) \subseteq X$, that is for

$$cl(X) = \left\{ \{gxg^{-1} \mid g \in S_n\} x \in X \right\},$$

similarly $cl(Y) \subseteq Y$. If X and Y distinct then $\emptyset = X \cap Y \supseteq cl(X) \bigcap cl(Y)$. □

Remark 4. The conjugacy class of the symmetric group S_n is $\bigcup_{i \in L} cl(i)$, and $|cl_{S_n}| = \left| \bigcup_{i \in L} cl(i) \right| = \sum_{i \in L} |cl(i)|$. The order of the union of disjoint sets is the sum of its orders.

Using a computer program named GAP, we evaluate the conjugacy classes for symmetric group S_n , $n = 3, 4, 5, \cdots, 9$ and we get

$$|cl_{S_3}| = 3, \quad |cl_{S_4}| = 5, \quad |cl_{S_5}| = 7, \quad |cl_{S_6}| = 11,$$

$$|cl_{S_7}| = 15, \quad |cl_{S_8}| = 22, \quad |cl_{S_9}| = 30.$$

To illustrate our method, we first begin to count the conjugacy classes for S_6 as an example:

The order of S_6 is $|S_6| = 6! = 720$.

The set of cycles length classes is $L = \{0, 2, 3, 4, 5, 6\}$.

To find $|cl_{S_6}|$ we will find $\left| \bigcup_{i \in L} cl(i) \right| = \sum_{i \in L} |cl(i)|$:

$i = 0$	the conjugacy class for the identity element is of length	0
$i = 2$	we have the following forms using cycles of length $i = 2$ or less to get sum ≤ 6	$2+0,\ 2+2,\ 2+2+2$
$i = 3$	we have the following forms using cycles of length $i = 3$ or less to get sum ≤ 6	$3+0,\ 3+2,\ 3+3$
$i = 4$	we have the following forms using cycles of length $i = 4$ or less to get sum ≤ 6	$4+0,\ 4+2$
$i = 5$	we have the following form using cycles of length $i = 5$ or less to get sum ≤ 6	$5+0$
$i = 6$	we have the following form using cycles of length $i = 6$ or less to get sum ≤ 6	$6+0$

Table 1

Now the total sum of all $|cl(i)|\ , i \in L$ is $|cl(0)| + |cl(2)| + |cl(3)| + |cl(4)| + |cl(5)| + |cl(6)| = 1 + 3 + 3 + 2 + 1 + 1 = 11$, so

$$|cl_{S_6}| = \underbrace{1}_{|cl(0)|} + \sum_{i=2}^{6} |cl(i)| = 11.$$

Let S_n be the symmetric group of order $n!$ and let i denote the cycle length. Then:

$$|cl_{S_n}| = 1 + \sum_{i=2}^{n} |cl(i)|. \tag{$*$}$$

From Table 1, one can see that we can use mixed lengths $3+2,\ 4+2$ for it give rise distinct conjugacy classes. The way we have to used such elements, is to get total lengths sum less than or equal n; it is clear that $4+2 \leq 6$ so it is one of our choices, but $4+3 = 7 \geq 6$ is omitted.

All what we need now is to describe the relation between $\sum_{i=2}^{n} |cl(i)|$ and n in terms of i and n.

In any symmetric group S_n, the set of cycles length is $L = \{0, 2, 3, \cdots, n\}$, and to find $|cl(i)|\ ,\ i \in L$, we need to count the number of choices we can get when we use a cycle of length i with all cycles j in L such that $j \leq i$ and the total sum $\leq n$. So we need to count the number of solutions M_i for the following inequality:

$$i + T_i \leq n,$$

where T_i is the total sum of cycles with length less than or equal i, for each $i \in L$. That is, let $n = 6$, then $L = \{0, 2, 3, 4, 5, 6\}$, and using the formula (*) we have $|cl_{S_6}| = 1 + \sum_{i=2}^{6} |cl(i)|$.

As seen in Table 1, $T_0 = 0$ then $M_0 = 1$, that is for $0 + T_0 \leq 6$ has a unique solution. Next, for $i = 2$, then $2 + T_2 \leq 6$ has three solutions $T_2 = 0,\ 2,\ 4$, and then $M_2 = 3$, similarly $M_3 = 3\ ,\ M_4 = 2\ ,\ M_5 = 1$ and $M_6 = 1$. Then $|cl_{S_6}| = 1 + \sum_{i=2}^{6} |cl(i)| = \sum_{i \in L} M_i = 11$.

This illustration can state the following theorem:

Theorem 1. *If G is the symmetric group S_n, then $|cl_G| = \sum_{i \in L} M_i$, where M_i is the number of solutions for*

$$i + \sum_{j=1}^{r} l \leq n \qquad \text{where } r = \left\lfloor \frac{n}{i} \right\rfloor, l \in \{0, 2, 3, \cdots, i\} \quad \text{and } i = 0, 2, 3, \cdots, n.$$

Here $\lfloor \frac{n}{i} \rfloor$ is the greatest integer less than or equal $\frac{n}{i}$, so r denotes the maximal number of cycles may used in each $|cl(i)|$.

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