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IRREDUCIBLE REPRESENTATIONS OF THE PURE BRAID GROUP P_n

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Abstract: We consider the complex specialization of Wada's representation of the pure braid group on n strings, namely $\phi : P_n \longrightarrow GL_n(C)$. Our main theorem is to find sufficient conditions under which the representation ϕ is irreducible.

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1. Introduction

Let B_n be the braid group on n strings. There are many known representations of B_n in the group $\mathrm{Aut}(F_n)$ of automorphisms of the free group F_n generated by $x_1, \ldots, x_n$. One of them is obtained by M. Wada, where the automorphism corresponding to σ_i takes $x_i \to {x_i}^2 x_{i+1}$, $x_{i+1} \to {x_{i+1}}^{-1} {x_i}^{-1} x_{i+1}$, and fixes all other free generators. In our work, we consider the complex specialization of Wada's representation applied to a free normal subgroup of the braid group, namely P_n, where $x_i \to t_i \in \mathbb{C}^*$. We show that the representation is irreducible if $t_i t_{i+1} \neq \pm 1$ for all $1 \leq i \leq n-1$.

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2. Definitions

Definition 1. [1] The Artin braid group, B_n, is the group generated by the generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ and the braid relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

for all $i, j = 1, 2, ..., n-1$ with $i - j \geq 2$, and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for $i = 1, ..., n-2$.

Definition 2. [2] The pure braid group, P_n, is defined as the kernel of the homomorphism $B_n \to S_n$, defined by $\sigma_i \to (i, i+1)$, $1 \leq i \leq n-1$. It has the following generators:

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2} \dots \sigma_{i+1}{\sigma_i}^2 \sigma_{i+1}^{-1} \dots \sigma_{j-2}^{-1} \sigma_{j-1}^{-1}, \;\; 1 \leq i < j \leq n.$$

Definition 3. [3] Let F_n be the free group of rank n, with free basis $x_1, x_2, ..., x_n$. The generators σ_i under Wada's representation are automorphisms of a free group F_n given by

$$x_i \longrightarrow x_i^2 x_{i+1},$$

$$x_{i+1} \longrightarrow x_{i+1}^{-1} x_i^{-1} x_{i+1},$$

$$x_j \longrightarrow x_j, j \neq i, i+1.$$

We will follow the idea suggested by J. Birman in [2] to define our Wada's representation of the pure braid group. Let F_n be the free group of rank n, with free basis $x_1, \dots, x_n$. The Jacobian matrix is defined as follows:

$$J(A_{i,j}) = \begin{pmatrix} D_1(A_{i,j}(x_1)) & \dots & D_n(A_{i,j}(x_1)) \\ \vdots & & \vdots \\ D_1(A_{i,j}(x_n)) & \dots & D_n(A_{i,j}(g_n)) \end{pmatrix},$$

where $D_j = \phi d_j$. Here d_j is the Fox derivative defined in [2] and $\phi(x_i) = t_i$ for $i = 1, \dots, n$.

Definition 4. A representation $G \longrightarrow GL(C^n)$ is called a reducible representation if there exists a nonzero proper subspace of C^n that is invariant under the action of the generators of G.

3. Sufficient Conditions for the Irreducibility of $\phi : P_n \longrightarrow GL_n(C)$

After computing the Jacobian matrices for the generators of the pure braid group under Wada's representation, we determine sufficient conditions under which the representation is irreducible. Under direct computations, we have that

$$A_{1,2} = \begin{pmatrix} 1+t_1+t_1^2t_2 & t_1^2+t_1^3t_2 & 0 & 0 & 0 & \dots & 0 \\ -t_1^{-1}t_2^{-1}-t_1^{-2}t_2^{-2} & -t_2^{-1}-t_2^{-2}t_1^{-1}+t_2^{-2}t_1^{-2} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

and

$$A_{1,3} = \begin{pmatrix} a & b & c & 0 & 0 & 0 & \dots & 0 \\ d & e & f & 0 & 0 & 0 & \dots & 0 \\ m & n & p & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

where

$a = 1 + t_1 + t_1^2t_2^2t_3$,
$b = t_1^2 + t_1^2t_2 + t_1^3t_2^2t_3 + t_1^3t_2^3t_3$,
$c = t_1^2t_2^2 + t_1^3t_2^4t_3$,
$d = -t_3^{-1}t_2^{-2}t_1^{-1} - t_3^{-2}t_2^{-4}t_1^{-2} + t_3^{-2}t_2^{-3}t_1^{-2} + t_3^{-1}t_2^{-1}t_1^{-1}$,
$e = -t_3^{-1}t_2^{-2} - t_3^{-2}t_2^{-4}t_1^{-1} + t_3^{-2}t_2^{-4}t_1^{-2} + t_3^{-2}t_2^{-2}t_1^{-1} + t_3^{-1}$,
$f = -t_3^{-1} - t_3^{-2}t_2^{-2}t_1^{-1} + t_3^{-2}t_2^{-1}t_1^{-1} + t_3^{-1}t_2$,
$m = -t_3^{-1}t_2^{-2}t_1^{-1} - t_3^{-2}t_2^{-4}t_1^{-2}$,
$n = -t_3^{-1}t_2^{-1} - t_3^{-1}t_2^{-2} - t_3^{-2}t_2^{-3}t_1^{-1} - t_3^{-2}t_2^{-4}t_1^{-1}$,
$p = -t_3^{-1} - t_3^{-2}t_2^{-2}t_1^{-1} + t_3^{-2}t_2^{-4}t_1^{-2}$.

We also have that

$$A_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 + t_2 + {t_2}^2 t_3 & {t_2}^2 + {t_2}^3 t_3 & 0 & 0 & 0 & \dots & 0 \\ 0 & -{t_3}^{-1}{t_2}^{-1} - {t_3}^{-2}{t_2}^{-2} & -{t_3}^{-1} - {t_3}^{-2}{t_2}^{-1} + {t_3}^{-2}{t_2}^{-2} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

For $1 \leq i \leq n-1$, we have an explicit form for the matrix $A_{i,i+1}$. More precisely, we have that

$$A_{i,i+1} = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & \alpha_i & \beta_i & 0 \\ 0 & \gamma_i & \delta_i & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix},$$

where

$$\begin{aligned} \alpha_i =& 1 + t_i + t_i^2 t_{i+1}, \\ \beta_i =& t_i^2 + t_i^3 t_{i+1}, \\ \gamma_i =& -t_{i+1}^{-1} t_i^{-1} - t_{i+1}^{-2} t_i^{-2}, \end{aligned}$$

and

$$\delta_i = -t_{i+1}^{-1} - t_{i+1}^{-2} t_i^{-1} + t_{i+1}^{-2} t_i^{-2}.$$

Now, we state and prove our main theorem.

Theorem 1. *If $t_i t_{i+1} \neq \pm 1$ for all $1 \leq i \leq n-1$, then the representation $\phi : P_n \longrightarrow GL_n(C)$ is irreducible.*

Proof. Let S be a nonzero invariant subspace of C^n under the representation ϕ. Then we have the following cases:

Case 1: Assume that $e_1 \in S$. It follows that $A_{1,2}e_1 = \alpha_1 e_1 + \gamma_1 e_2 \in S$, where $\gamma_1 \neq 0$. This implies that $e_2 \in S$. By applying $A_{2,3}$ on e_2, we get that $e_3 \in S$ and so we proceed in this manner to prove that $S = C^n$.

Case 2: Assume that $e_i \in S$ for some $i > 1$. It follows that $A_{i-1,i}e_i = \beta_{i-1}e_{i-1} + \delta_{i-1}e_i \in S$, where $\beta_{i-1} \neq 0$. This implies that $e_{i-1} \in S$. Proceeding in this way, we get that $e_1 \in S$. By Case 1, it follows that all e_i's are in S and so $S = C^n$.

Case 3: Assume that $e_i + ae_{i+1} \in S$ for some $1 \leq i < n$ and $a \neq 0$. If $u = e_1 + ae_2 \in S$, then $A_{1,2}u = (1 + t_1 + t_1^2t_2 + a(t_1^2 + t_1^3t_2))e_1 + (a(\frac{1}{t_1^2t_2^2} - \frac{1}{t_1t_2^2} - \frac{1}{t2}) - \frac{1}{t_1^2t_2^2} - \frac{1}{t_1t_2})e_2 \in S$. We then have that $A_{1,2}u - (1+t_1+t_1^2t_2+a(t_1^2+t_1^3t_2))u =$

$$-\frac{(1 + t_1t_2)(1 + a^2t_1^4t_2^2 + a(-1 + t_1 + t_1t_2 + t_1^3t_2^2))}{t_1^2t_2^2}e_2 \in S.$$

- If

$$a \neq \frac{1 - t_1 - t_1t_2 - t_1^3t_2^2 \pm \sqrt{-4t_1^4t_2^2 + (-1 + t_1 + t_1t_2 + t_1^3t_2^2)^2}}{2t_1^4t_2^2},$$

then

$$-\frac{(1 + t_1t_2)(1 + a^2t_1^4t_2^2 + a(-1 + t_1 + t_1t_2 + t_1^3t_2^2))}{t_1^2t_2^2} \neq 0,$$

which implies that $e_2 \in S$, and so $S = C^n$.

- If

$$a = \frac{1 - t_1 - t_1t_2 - t_1^3t_2^2 - \sqrt{-4t_1^4t_2^2 + (-1 + t_1 + t_1t_2 + t_1^3t_2^2)^2}}{2t_1^4t_2^2},$$

we let $v = A_{2,3}u$ and $w = A_{1,2}A_{2,3}u$. Then we have that $w - v - \lambda\, u \in S$, where λ is the first component of the vector $w - v$. This means that

$$-\frac{(1 + t_1t_2)(-1 - t_1 + t_1t_2 + t_1^3t_2^2 + \sqrt{-4t_1^4t_2^2 + (-1 + t_1 + t_1t_2 + t_1^3t_2^2)^2}(1 + t_2t_3)}{2t_1^3t_2}$$
$$e_2 \in S,$$

where

$$-\frac{(1 + t_1t_2)(-1 - t_1 + t_1t_2 + t_1^3t_2^2 + \sqrt{-4t_1^4t_2^2 + (-1 + t_1 + t_1t_2 + t_1^3t_2^2)^2}(1 + t_2t_3)}{2t_1^3t_2}$$
$$= 0,$$

only if $t_1t_2 = 1$ or $t_1t_2 = -1$ or $t_2t_3 = -1$. Since this is not the case according to our hypothesis, we conclude that $e_2 \in S$.

The same argument holds for

$$a = \frac{1 - t_1 - t_1t_2 - t_1^3t_2^2 + \sqrt{-4t_1^4t_2^2 + (-1 + t_1 + t_1t_2 + t_1^3t_2^2)^2}}{2t_1^4t_2^2}.$$

Now, in general, if $u_i = e_i + ae_{i+1} \in S$ for some i, then $A_{i-1,i}u_i = \beta_{i-1}e_{i-1} + \delta_{i-1}e_i + ae_{i+1} \in S$, which implies that $v_i = A_{i-1,i}u_i - u_i = \beta_{i-1}e_{i-1} + (\delta_{i-1} - 1)e_i \in S$. Applying $A_{i-2,i-1}$, then $A_{i-3,i-2}$, etc. $\ldots$, we get that a linear combination of e_1 and e_2 is an element of S and so we are done.

Case 4: Assume that $e_{i_1} + a_{i_2}e_{i_2} + ... + a_{i_k}e_{i_k} \in S$, where $a_{i_j} \neq 0$ and $i_1 < i_2 < .... < i_k \quad (j = 1, ..., k)$.

Suppose that u=$e_{i_1} + a_{i_2}e_{i_2} + ... + a_{i_k}e_{i_k} \in S$. We have that either $i_2 > i_1 + 1$ or $i_2 = i_1 + 1$.

• If $i_2 > i_1 + 1$ then $A_{i_1,i_1+1}u = \alpha_{i_1}e_{i_1} + \gamma_{i_1}e_{i_1+1} + a_{i_2}e_{i_2} + ... + a_{i_k}e_{i_k} \in S$. This implies that $A_{i_1,i_1+1}u - u = (\alpha_{i_1} - 1)e_{i_1} + \gamma_{i_1}e_{i_1+1} \in S$, and so $S = C^n$ by Case 3.

• If $i_2 = i_1 + 1$ then $A_{i_1,i_1+1}u = (\alpha_{i_1} + \beta_{i_1}a_{i_2})e_{i_1} + (\gamma_{i_1} + \delta_{i_1}a_{i_2})e_{i_2} + a_{i_3}e_{i_3} + ... + a_{i_k}e_{i_k} \in S$. This implies that $A_{i_1,i_1+1}u - u = (\alpha_{i_1} + \beta_{i_1}a_{i_2} - 1)e_{i_1} + (\gamma_{i_1} + \delta_{i_1}a_{i_2} - a_{i_2})e_{i_2} \in S$ and so $S = C^n$. □

References

[1] E. Artin, *The Collected Papers of Emil Artin*, Addison-Wesley Publishing Company, Inc. (1965).

[2] J.S. Birman, *Braids, Links and Mapping Class Groups*, Annals of Mathematical Studies, **82**, Princeton University Press, New Jersey (1975).

[3] M. Wada, Group invariants of links, *Topology*, **31**, No. 2 (1992), 399-406.