International Journal of Applied Mathematics

Volume 25 No. 5 2012, 729-737

EXISTENCE OF VISCOSITY SOLUTION TO SYSTEMS OF ONE-DIMENSIONAL EULER EQUATIONS WITH A SOURCE

Juan Carlos Hernández R.
Department of Mathematics
Universidad Nacional de Colombia
Kra. 30 No. 45-03, Oficina 308, Bogotá, D.C., COLOMBIA
e-mail: jchernandezri@unal.edu.co

Abstract: We study the existence of local smooth viscosity solution for the Cauchy problem of the diffusion system related to a system of one-dimensional Euler equations with source, then based on the local existence and an a-priori L^∞ estimate for the solution, we get the existence of global viscosity solution. Also, using the maximum principle and the invariant region theory, we obtain the existence of global solution for some source terms.

AMS Subject Classification: 35D40, 35B45
Key Words: one-dimensional Euler equations, viscosity solution, source, maximum principle, invariant region

1. Introduction

The system

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + \left(\frac{1}{2}u^2 + p(\rho)\right)_x = 0\,, \end{cases} \tag{1}$$

was first derived by S. Earnshaw in [6] for isentropic flow and is also called the Euler equations of one-dimensional, here ρ and u denote, respectively, the density and velocity of the fluid, and $p(\rho)$ is the pressure. For smooth solutions, (1) is equivalent to the classical system of Euler equations in the isentropic

Received: September 13, 2012

case, this consists of the system of isentropic gas dynamics in Euler coordinates ([4],[10])

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + \big(\rho u^2 + p(\rho)\big)_x = 0\,. \end{cases} \tag{2}$$

The Cauchy problem for (1) has been studied in many different papers, for example [1, 2, 5, 6, 7, 9]. The Cauchy problem for (1) when the initial datum satisfies

$$\begin{cases} \rho(x,0) = \rho_0(x),\ u(x,0) = u_0(x) \in L^\infty(\mathbb{R}), \\ \rho_0(x) \geq 0\,, \end{cases} \tag{3}$$

was also studied in [8] and [9] for two special cases of the function $p(\rho)$ (which are very interesting from the mathematical point of view), for it the author first considered the existence of solution for the Cauchy problem of the diffusive system associated with the system (1)

$$\begin{cases} \rho_t + (\rho u)_x = \epsilon\rho_{xx} \\ u_t + \big(\frac{1}{2}u^2 + p(\rho)\big)_x = \epsilon u_{xx}\,, \end{cases} \tag{4}$$

with initial data (3).

In this paper, we consider the Cauchy problem

$$\begin{cases} \rho_t + (\rho u)_x + g_1(\rho,u) = \epsilon\rho_{xx} \\ u_t + \big(\frac{1}{2}u^2 + p(\rho)\big)_x + g_2(\rho,u) = \epsilon u_{xx} \end{cases} \tag{5}$$

with initial data

$$\big(\rho(x,0), u(x,0)\big) = \big(\rho_0(x), u_0(x)\big) \tag{6}$$

where $\epsilon > 0$ is a constant called diffusion coefficient or viscosity parameter, $g_i(\rho,u)$, $i = 1,2$ are locally Lipschitz continuous functions, $p(\rho) \in C^1(\mathbb{R})$ and $\rho_0(x), u_0(x) \in L^\infty(\mathbb{R})$. The system (5) is the diffusion system associated to the system of one-dimensional Euler equations ([4],[10]) with source.

This paper is organized as follows. In Section 2, we show the existence of a unique local smooth viscosity solution for the Cauchy problem (5)-(6), which proof follows a classical scheme in the standard theory of parabolic systems. In Section 3, we get the existence of global solution for some source terms.

2. Existence of Smooth Viscosity Solution

The local existence and regularity of solution can be obtained for the Cauchy problem (5)-(6) by applying the Banach fixed point theorem to an integral representation for a solution, where the local time depends only on the L^∞ norm of the initial data.

Theorem 2.1. *Assume that* $||\rho_0(x)||_{L^\infty(\mathbb{R})}, ||u_0(x)||_{L^\infty(\mathbb{R})} \leq N$, *then the Cauchy problem (5)-(6) has a unique solution* $\big(\rho^\epsilon(x,t), u^\epsilon(x,t)\big) \in C^\infty(\mathbb{R} \times (0, \mathfrak{t}_0))$ *for a small* $\mathfrak{t}_0$ *which depends only on the* L^∞ *norm of the initial data. Moreover*

$$||\rho^\epsilon(x,t)||_{L^\infty(\mathbb{R}\times[0,\mathfrak{t}_0])} \leq 2N, \quad ||u^\epsilon(x,t)||_{L^\infty(\mathbb{R}\times[0,\mathfrak{t}_0])} \leq 2N. \tag{7}$$

Proof. Let $K(x,t)$ be a fundamental solution associated to the operator $\frac{\partial}{\partial t} - \epsilon \frac{\partial^2}{\partial x^2}$, i.e.,

$$K(x,t) = \frac{1}{\sqrt{4\pi\epsilon t}} e^{-\frac{x^2}{4\epsilon t}}. \tag{8}$$

A solution of (5)-(6) satisfies the following integral equations ([3])

$$\rho(t) = K(t) * \rho_0 - \int_0^t \Big(\rho(s)u(s) * K_x(t-s) + g_1\big(\rho(s), u(s)\big) * K(t-s)\Big)\, ds\,, \tag{9}$$

$$u(t) = K(t) * u_0 - \int_0^t \Big(\Big(\tfrac{1}{2}u^2(s) + p\big(\rho(s)\big)\Big) * K_x(t-s) + g_1\big(\rho(s), u(s)\big) * K(t-s)\Big)\, ds, \tag{10}$$

where $*$ denotes the convolution and for a function $u(x,t)$ we denote by $u(t)$ a function of $x \in \mathbb{R}$ defined by $u(t)(x) = u(x,t)$.

For any $\mathfrak{t} > 0$, we define the set of functions $\mathfrak{B}_\mathfrak{t}$ by

$$\mathfrak{B}_\mathfrak{t} = \Big\{\big(\rho(t), u(t)\big) \in C(\mathbb{R} \times (0, \mathfrak{t})) : ||\rho(t)||_{L^\infty(\mathbb{R}\times[0,\mathfrak{t}])} \leq 2N, ||u(t)||_{L^\infty(\mathbb{R}\times[0,\mathfrak{t}])} \leq 2N\Big\}$$

and the metric d on $\mathfrak{B}_{\mathfrak{t}}$ by

$$d\Big(\big(\rho(t),u(t)\big),\big(\varphi(t),v(t)\big)\Big) = ||\rho(t)-\varphi(t)||_{L^\infty(\mathbb{R}\times[0,\mathfrak{t}])} + ||u(t)-v(t)||_{L^\infty(\mathbb{R}\times[0,\mathfrak{t}])},$$

then the metric space $\boldsymbol{\mathfrak{B}}_{\mathbf{t}} = \big(\mathfrak{B}_{\mathfrak{t}}, d\big)$ is complete. The fact that the functions $f_1(\rho,u) = \rho u$, $f_2(\rho,u) = \frac{1}{2}u^2 + p(\rho)$, $g_1(\rho,u)$ and $g_2(\rho,u)$ are locally Lipschitz continuous, means that for any $(\rho,u),(\varphi,v) \in \mathfrak{B}_{\mathfrak{t}}$, there exist positive constants L, M such that

$$\big|f_i(\rho,u)\big| \le L, \quad \big|g_i(\rho,u)\big| \le L, \quad i=1,2, \tag{11}$$

$$\big|f_i(\varphi,v) - f_i(\rho,u)\big| \le M\big(|\varphi-\rho| + |v-u|\big), \quad i=1,2, \tag{12}$$

$$\big|g_i(\varphi,v) - g_i(\rho,u)\big| \le M\big(|\varphi-\rho| + |v-u|\big), \quad i=1,2. \tag{13}$$

Let $\boldsymbol{\mathcal{L}}$ be the operator defined by

$$\boldsymbol{\mathcal{L}}\big(\rho(t),u(t)\big) = \Big(\mathcal{L}_1\big(\rho(t),u(t)\big), \mathcal{L}_2\big(\rho(t),u(t)\big)\Big), \quad \big(\rho(t),u(t)\big) \in \mathfrak{B}_{\mathfrak{t}},$$

where

$$\mathcal{L}_1\big(\rho(t),u(t)\big) = K(t)*\rho_0 - \int_0^t \Big(\rho(s)u(s)*K_x(t-s) + g_1\big(\rho(s),u(s)\big)*K(t-s)\Big)\,ds\,,$$

$$\mathcal{L}_2\big(\rho(t),u(t)\big) = K(t)*u_0 - \int_0^t \Big(\Big(\tfrac{1}{2}u^2(s) + p\big(\rho(s)\big)\Big)*K_x(t-s) + g_1\big(\rho(s),u(s)\big)*K(t-s)\Big)\,ds\,.$$

From (11) and using that

$$\int_{-\infty}^{+\infty} K(x-\xi,t)\,d\xi = 1, \quad \int_0^t \int_{-\infty}^{+\infty} -K_x(x-\xi,t-s)\,d\xi\,ds = 2\sqrt{\frac{t}{\pi\epsilon}}\,,$$

$t>0$, we obtain

$$\Big|\mathcal{L}_i\big(\rho(t),u(t)\big)\Big| \le N + L\left(2\sqrt{\frac{\mathfrak{t}}{\pi\epsilon}} + \mathfrak{t}\right), \quad i=1,2, \tag{14}$$

also from (12)-(13) it follows

$$\Big|\mathcal{L}_i\big(\varphi(t), v(t)\big) - \mathcal{L}_i\big(\rho(t), u(t)\big)\Big| \\ \leq M\left(2\sqrt{\frac{\mathfrak{t}}{\pi\epsilon}} + \mathfrak{t}\right) d\Big(\big(\rho(t), u(t)\big), \big(\varphi(t), v(t)\big)\Big),$$

$i = 1, 2$. Thus

$$d\Big(\boldsymbol{\mathcal{L}}\big(\varphi(t), v(t)\big), \boldsymbol{\mathcal{L}}\big(\rho(t), u(t)\big)\Big) \\ \leq 2M\left(2\sqrt{\frac{\mathfrak{t}}{\pi\epsilon}} + \mathfrak{t}\right) d\Big(\big(\rho(t), u(t)\big), \big(\varphi(t), v(t)\big)\Big). \quad (15)$$

If we choose $\mathfrak{t} = \mathfrak{t}_0$ such that

$$L\left(2\sqrt{\frac{\mathfrak{t}_0}{\pi\epsilon}} + \mathfrak{t}_0\right) \leq N, \quad 2M\left(2\sqrt{\frac{\mathfrak{t}_0}{\pi\epsilon}} + \mathfrak{t}_0\right) < 1,$$

then (14) and (15) prove respectively that $\boldsymbol{\mathcal{L}}\big(\rho(t), u(t)\big) \in \mathfrak{B}_{\mathfrak{t}_0}$ and $\boldsymbol{\mathcal{L}} : \boldsymbol{\mathfrak{B}}_{\mathfrak{t}_0} \to \boldsymbol{\mathfrak{B}}_{\mathfrak{t}_0}$ is a contraction. By the Banach fixed point theorem, $\boldsymbol{\mathcal{L}}$ has a unique fixed point $\big(\rho^\epsilon, u^\epsilon\big) \in \mathfrak{B}_{\mathfrak{t}_0}$. From this fact and from (9)-(10) if follows that the Cauchy problem (5)-(6) has a unique local solution $\big(\rho^\epsilon(x,t), u^\epsilon(x,t)\big) \in C^\infty(\mathbb{R} \times (0, \mathfrak{t}_0))$ satisfying (7). □

If the problem (5)-(6) satisfies an a-priori estimate in L^∞, by using the local existence, we have the following global existence result about the Cauchy problem (5)-(6).

Theorem 2.2. *If $\big(\rho^\epsilon(x,t), u^\epsilon(x,t)\big)$ has an a-priori estimate*

$$||\rho^\epsilon(x,t)||_{L^\infty(\mathbb{R}\times[0,T])} \leq N(T), \quad ||u^\epsilon(x,t)||_{L^\infty(\mathbb{R}\times[0,T])} \leq N(T), \quad (16)$$

then the solution $\big(\rho^\epsilon(x,t), u^\epsilon(x,t)\big)$ of (5)-(6) exist on $\mathbb{R} \times [0, T]$.

Proof. Since $\big(\rho^\epsilon(x,t), u^\epsilon(x,t)\big)$ has the a-priori estimate (16), then

$$||\rho_0(x)||_{L^\infty(\mathbb{R})} \leq N(T), \quad ||u_0(x)||_{L^\infty(\mathbb{R})} \leq N(T).$$

By Theorem 2.1, there is a small $\mathfrak{t}_0 > 0$ which depends only on $N(T)$ such that (5)-(6) has a unique solution $\big(\rho^\epsilon(x,t), u^\epsilon(x,t)\big)$ on $\mathbb{R} \times [0, \mathfrak{t}_0]$, satisfying (16). We can use $\big(\rho^\epsilon(x, \mathfrak{t}_0), u^\epsilon(x, \mathfrak{t}_0)\big)$ as new initial data, from (16) we have

$$||\rho^\epsilon(x, \mathfrak{t}_0)||_{L^\infty(\mathbb{R})} \leq N(T), \quad ||u^\epsilon(x, \mathfrak{t}_0)||_{L^\infty(\mathbb{R})} \leq N(T),$$

then the solution also exist on $\mathbb{R} \times [\mathfrak{t}_0, 2\mathfrak{t}_0]$. Therefore, the local time $\mathfrak{t}_0$ can be extended to T step by step since the step time depends only on $N(T)$, thus we obtain a solution on $\mathbb{R} \times [0, T]$. $\square$

According to the previous proof, one has the next result.

Corollary 2.1. *If* $\big(\rho^\epsilon(x,t), u^\epsilon(x,t)\big)$ *has an a-priori estimate*

$$||\rho^\epsilon(x,t)||_{L^\infty(\mathbb{R}\times[0,+\infty))} \leq N, \quad ||u^\epsilon(x,t)||_{L^\infty(\mathbb{R}\times[0,+\infty))} \leq N, \tag{17}$$

then the solution of (5)-(6) exist on $\mathbb{R} \times [0, +\infty)$.

The solution obtained in Theorems 2.1, 2.2 or Corollary 2.1 is called *viscosity solution*. This solution can be used to get a global weak solution or generalized solution for (1)-(6), using the vanishing method with the help of the theory of compensated compactness.

3. Existence of Global Viscosity Solution for Some Source Terms

We consider the Cauchy problem (5)-(6). From now on, we restrict our interest to solution corresponding to initial data values such that $\rho_0(x) \geq 0$, in this case we will prove the existence of global viscosity solution under certain conditions on the function p and on the source terms g_i, $i = 1, 2$.

Let F be the mapping defined by

$$\begin{aligned} F : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (\rho, u) &\rightarrow \Big(\rho u, \tfrac{1}{2}u^2 + p(\rho)\Big), \end{aligned}$$

the Jacobian matrix of F is

$$dF_{(\rho,u)} = \begin{pmatrix} u & \rho \\ p'(\rho) & u \end{pmatrix},$$

thus the eigenvalues of system (1) are

$$\lambda_1 = u - \sqrt{\rho p'(\rho)}\,, \quad \lambda_2 = u + \sqrt{\rho p'(\rho)}\,.$$

The Riemann invariants of (1) are functions $w = w(\rho, u)$ and $z = z(\rho, u)$ satisfying the equations

$$\nabla w dF = \lambda_2 w, \quad \nabla z dF = \lambda_1 z. \tag{18}$$

One solution of (18) is

$$w(\rho,u)=u+\int_0^{\rho}\sqrt{\frac{p'(s)}{s}}\,ds\,,\quad z(\rho,u)=u-\int_0^{\rho}\sqrt{\frac{p'(s)}{s}}\,ds\,. \tag{19}$$

Theorem 3.1. *Assume that $\rho_0(x)\geq 0$; $p(\rho)\in C^2(0,\infty)$; $p'(\rho)>0$, $\rho p''(\rho)-p'(\rho)\geq 0$ for $\rho>0$ and*

$$\int_0^{c}\sqrt{\frac{p'(s)}{s}}\,ds<\infty,\ \ \int_c^{\infty}\sqrt{\frac{p'(s)}{s}}\,ds=\infty,\ \ \forall c>0. \tag{20}$$

Also suppose that $g_1(\rho,u),g_2(\rho,u)$ satisfy the following inequalities

$$-\sqrt{\frac{p'(\rho)}{\rho}}g_1(\rho,u)\leq g_2(\rho,u)\leq\sqrt{\frac{p'(\rho)}{\rho}}g_1(\rho,u)\ \ \text{for } \rho>0 \tag{21}$$

and $g_1(\rho,u)=\rho h(\rho,u)$, where $h(\rho,u)$ is continuous function. Then the Cauchy problem (5)-(6) has a unique solution $\big(\rho^{\epsilon}(x,t),u^{\epsilon}(x,t)\big)$ on $\mathbb{R}\times[0,+\infty)$ satisfying

$$0\leq\rho^{\epsilon}(x,t)\leq M,\quad |u^{\epsilon}(x,t)|\leq M, \tag{22}$$

where M is a positive constant.

Proof. To prove the existence of global viscosity solution for the Cauchy problem (5)-(6), according to Corollary 2.1, we only need to prove the a-priori estimates in (22).

We multiply system (5) by (w_ρ,w_u) and (z_ρ,z_u), respectively, where w,z are given by (19), to obtain

$$w_t+\lambda_2 w_x=\epsilon w_{xx}-\frac{\epsilon\rho_x^2}{2\rho^2}\sqrt{\frac{\rho}{p'(\rho)}}\big(\rho p''(\rho)-p'(\rho)\big)-(g_1w_\rho+g_2w_u) \tag{23}$$

and

$$z_t+\lambda_1 z_x=\epsilon z_{xx}+\frac{\epsilon\rho_x^2}{2\rho^2}\sqrt{\frac{\rho}{p'(\rho)}}\big(\rho p''(\rho)-p'(\rho)\big)-(g_1z_\rho+g_2z_u). \tag{24}$$

Inserting the inequalities (21) and the assumptions on $p(\rho)$ in the equalities (23)-(24), we obtain

$$w_t+\lambda_2 w_x\leq\epsilon w_{xx} \tag{25}$$

and

$$z_t+\lambda_1 z_x\geq\epsilon z_{xx}. \tag{26}$$

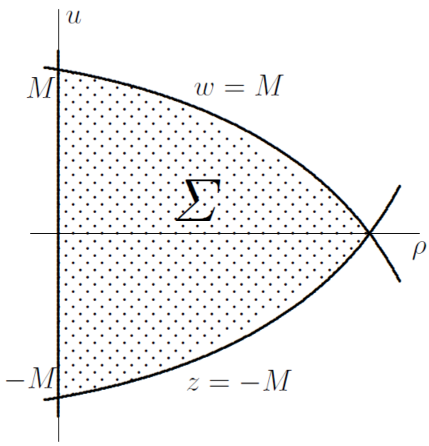

Figure 3.1

Applying the maximum principle to (25) and (26), we have that $w(\rho^\epsilon, u^\epsilon) \leq N$ and $z(\rho^\epsilon, u^\epsilon) \geq -N$ for a suitable large constant positive N depending only on L^∞ bound of the initial data. Using the first equation in (5) and also that $\rho_0(x) \geq 0$, we get $\rho^\epsilon \geq 0$. It follows from the invariant region theory ([4]) that the region

$$\Sigma = \left\{(\rho, u) \mid w(\rho, u) \leq N, z(\rho, u) \geq -N, \rho \geq 0\right\}$$

is a bounded invariant region (see Figure 3.1). Thus we obtain the estimates in (22) for a suitable positive constant M, since we assume (20). $\square$

Remark 3.1. There are many functions g_1 y g_2 satisfying the assumptions of Theorem 3.1. For example

$$g_1(\rho, u) = \alpha \rho u^2 \,, \quad g_2(\rho, u) = \beta \sqrt{\rho p'(\rho)} u^2 \,,$$

where α, β are real constants such that $|\beta| \leq \alpha$.

References

[1] A. Bressan, The unique limit of the Glimm scheme, *Arch. Rat. Mech. Anal.*, **130** (1995), 205-230.

[2] G.B. Whitham, *Linear and Nonlinear Waves*, John Wiley and Sons, New York (1973).

[3] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, Berlin-Heidelberg-New York (1983).

[4] K.N. Chueh, C.C. Conley, J.A. Smoller, Positive invariant regions for systems of nonlinear equations, *Indiana Univ. Math. J.*, **26** (1977), 373-392.

[5] R.J. Diperna, Global solutions to a class of nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.*, **26** (1973), 1-28.

[6] S. Earnshaw, On the mathematical theory of sound, *Philos. Trans. of the Royal Society of London*, **150** (1860), 1150-1154.

[7] Y.G. Lu, Convergence of viscosity solutions to a nonstrictly hyperbolic system, In: *Advances in Nonlinear Differential Equations and Related Areas, World Scientific* (1998), 250-266.

[8] Y.G. Lu, Convergence of the viscosity method for nonstrictly hyperbolic conservation laws, *Commun. Math. Phys.*, **150** (1992), 59-64.

[9] Y.G. Lu, Existence of global entropy solutions to a nonstrictly hyperbolic system, *Arch. Rat. Mech. Anal.*, **178** (2005), 287-299.

[10] Y.G. Lu, *Hyperbolic Conservations Laws and the Compensated Compactness Method*, **128**, Chapman and Hall, New York (2002).