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STATIONARY DISTRIBUTION OF FUZZY MARKOV CHAINS USING DIFFERENT QUASI-RANDOM SEQUENCES

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Abstract: A general definition of fuzzy Markov chains over a finite state space is introduced. It has been pointed out that unlike to the classical Markov chains recurrency does not imply ergodicity in the case of the fuzzy Markov chains. However, if a fuzzy Markov chain is ergodic then the rows of its limiting transition matrix are equal to the greatest eigen fuzzy set of the fuzzy relation associated with the chain. We make use of some quasi-random sequences for generating elements of fuzzy transition matrix.

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1. Introduction

The fuzzy Markov chains have a potential application in fuzzy Markov algorithms proposed by Zadeh in [9]. Avrachenkov and Sanchez [2] explored fuzzy Markov chains with a transition possibility measure and a general state space. Also, Kalenatic [7] presented a simulation study on fuzzy Markov chains to identify some characteristics about their behavior, based on matrix analysis.

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Through experimental evidence it is observed that some fuzzy Markov chains have an ergodic behavior. In this paper, the greatest eigen fuzzy set method is implemented, for obtaining the stationary distribution of ergodic fuzzy Markov chains. Further, we improved result of simulation using some quasi-random sequences such as Halton sequences and Sobol' sequences.

2. Basic Definitions of Fuzzy Markov Chains

Such as in the classical Markov processes analysis, the definition of a Fuzzy Markov Chain is based on a squared relational matrix that represents the possibility that a discrete state at instant t becomes into any state at next instant $t+1$ as follows:

$$P(X^{(t)} = s|X^{(t-1)} = x^{(t-1)}). \tag{1}$$

Here, $P(X^{(t)})$ is a fuzzy distribution of the process characterized by a membership function.

Definition 1. $S = \{1, 2, \dots, n\}$. A finite fuzzy set for a fuzzy distribution on S is defined by a mapping x from S to $[0, 1]$ represented by a vector $x = \{x_1, x_2, \dots, x_n\}$, with $0 \leq x_i \leq 1$, $i \in S$.

In this definition, x_i is the membership degree that a state i has regarding a fuzzy set S, $i \in S$ with cardinality m, $\mathcal{C}\ (S) = m$. All relations and compositions are defined by fuzzy sets theory since are useful tools to find a fuzzy stationary distribution.

Now, a fuzzy relational matrix P is defined in a metric space $S \times S$ by a matrix $\{p_{ij}\}_{i,j=1}^{m}$ with $0 \leq p_{ij} \leq 1$, $i, j \in S$. The complete set of all fuzzy sets is denoted by $\mathcal{F}\ (S)$ where $\mathcal{C}\ (S) = m$.

This fuzzy matrix P allows to define all relations among the m states of the fuzzy Markov chain at each time instant t, as follows.

Definition 2. At each instant t, $t = 1, 2, \dots, n$, the state of system is described by the fuzzy set $x^{(t)} \in \mathcal{F}\ (S)$. The transition law of a fuzzy Markov chain is given by the fuzzy relational matrix P at instant t, $t = 1, 2, \dots, n$, as follows:

$$x_j^{(t+1)} = \max_{i \in S}\{x_j^{(t)} \wedge p_{ij}\}, \qquad j \in S, \tag{2}$$

$$x^{(t+1)} = x^{(t)} \circ P, \tag{3}$$

where i and j, $i, j = 1, 2, \dots, m$ are the initial and final states of the transition and $x^{(0)}$ is the initial distribution.

Definition 3. (Stationary Distribution) Let the powers of the fuzzy transition matrix P converge in τ steps to a non-periodic solution, then the associated fuzzy Markov chain is called aperiodic fuzzy Markov chain and $P^* = P^\tau$ is its stationary fuzzy transition matrix.

Definition 4. (Strong Ergodicity and Week Ergodicity) A fuzzy Markov chain is called strong Ergodic if it is aperiodic and its stationary transition matrix has identical rows.

A fuzzy Markov chain is called weakly Ergodic if it is aperiodic and its stationary transition matrix is stable with no identical rows.

3. Eigen Fuzzy Set

Sanchez in [2], [8] defined its stationary distribution by its eigen fuzzy set which is defined next:

Definition 5. Let P be a fuzzy relation in a given matrix form. Then x is called an Eigen Fuzzy Set of P, iff:

$$x \circ P = x. \tag{4}$$

Definition 6. The fuzzy set $x \in \mathcal{F}\ (S)$ is contained in the fuzzy set $y \in \mathcal{F}\ (S)$, this is, $(x \subseteq y)$, iff $x_i \leq y_i$ for all $i \in S$.

Definition 7. Let χ be the set of eigen fuzzy sets of the fuzzy relation P. Namely:

$$\chi = \{x \in \mathcal{F}(S) | x \circ P = x\}. \tag{5}$$

The elements of χ are invariants of P according to the $\circ - (\max - \min)$ composition. Then, if there exists $\breve{x} \in \mathcal{F}\ (S)$ such that $x \subseteq \breve{x}$ for any $x \in \chi$, it is called the Greatest Eigen Fuzzy Set of the relation P.

Theorem 1. *There exists $m \in \{1, 2, ..., n\}$ such that $(\breve{x} = x^m)$ is the greatest element in χ. Moreover, $x^0 \subseteq \breve{x} \subseteq x^m$.*

Theorem 2.

$$\max_{i \in s} p_{ij}^k = (x^1 \circ P^{k-1}) = x_j^k, \quad j = \overline{1, n}, \quad for\ all\ \ k > 0. \tag{6}$$

One has the method III proposed by Sanchez in [8]:

- Determine first x^1 with the elements corresponding to the greatest element in each column of P.

- Compute $P^2 = P \circ P$ and determine the greatest elements in each column of P^2: they give x^2, according to Theorem 2, with $k = 2$: $\max_{i \in s} p^2_{ij} = x^2_j, j = \overline{1, 5.}$
- Compare with : if they are different, compute $P^3 = P^2 \circ P$ to get x^3 according to $\max_(i \in s) P^3_{ij} = x^3_j, j = \overline{1, 5}.$
- Compare x^3 with x^2: if they are different, compute $P^4 = P^3 \circ P$ to get x^4 , etc.

  Stop when it is found m such that $x^{m+1} = x^m$, that is $x^{m+1} = x^m \circ P$.

Theorem 3. *Suppose that the fuzzy Markov chain with transition matrix P is ergodic. Namely, it is aperiodic and the limiting transition matrix $P^* = P^\tau$ has identical rows. Then these rows are equal to $\breve{x}$, the greatest eigen fuzzy set of the fuzzy relation defined by P, see [2].*

4. Quasi-Random Sequences

The generation of quasi-random numbers is also at the heart of many standard statistical methods. In the next two subsections, the generation of Halton and Sobol' sequences is briefly explained. We discuss methods for generation of sequences of quasi-random numbers that simulate a uniform distribution over the unit interval $(0, 1)$ and use them for generating elements of fuzzy transition matrix.

4.1. Halton Sequences

Let p be a fixed prime number. Then any positive integer r can be uniquely written as its $p - adic$ expansion in the form

$$r = \sum_{i=0}^{m} a_i p^i, \quad a_i \in \{0, ..., p - 1\}, \quad i = 0, ..., m. \tag{7}$$

The rth number of the one-dimensional Halton sequence is defined by

$$y_r = \sum_{i=0}^{m} \frac{a_i}{p^{i+1}}. \tag{8}$$

The d-dimensional Halton sequence is generated taking d different prime numbers (usually the first d) and putting together the resulting d one-dimensional

$p=2$			$p=3$		
r	$\sum_{i=0}^{m} a_i p^i$	y_r	r	$\sum_{i=0}^{m} a_i p^i$	y_r
1	$1 \times 2^0 + 0 \times 2^1$	0.500	1	1×3^0	0.333
2	$0 \times 2^0 + 1 \times 2^1$	0.250	2	2×3^0	0.667
3	$1 \times 2^0 + 1 \times 2^1$	0.750	3	$0 \times 3^0 + 1 \times 3^1$	0.111
4	$0 \times 2^0 + 0 \times 2^1 + 1 \times 2^2$	0.125	4	$1 \times 3^0 + 1 \times 3^1$	0.444
5	$1 \times 2^0 + 0 \times 2^1 + 1 \times 2^2$	0.625	5	$2 \times 3^0 + 1 \times 3^1$	0.778

Table 1: First five Halton points using the primes $p = 2, 3$

sequences. Table 1 shows how to obtain the first five points using the values $p = 2, 3$. Note that by construction, all the resulting Halton points y_r lie in the interval $(0, 1)$. More details can be found in Halton [6].

4.2. Sobol' Sequences

Let

$$\nu_i = m_i 2^{-i}, \quad i = 1, 2, ..., \tag{9}$$

where m_i are odd positive integers chosen using the recursion

$$m_i = 2b_1 m_{i-1} \oplus 2^2 b_2 m_{i-2} \oplus ... \oplus 2^i b_i m_0, \tag{10}$$

according to a primitive polynomial

$$P(z) = z^p + b_1 m_{i-2} z^{p-1} + ... + b_{p-1} z + 1, \tag{11}$$

and $\oplus$ is the addition using binary arithmetic. The rth number of the one-dimensional Sobol' sequence is defined by

$$y_r = a_1 \nu_1 \oplus a_2 \nu_2 \oplus ..., \tag{12}$$

where $a_1, a_2, ...$ is the binary representation of r(see Eq. (7)). Antonve and Saleeve (1979) improved Sobol's original algorithm and proposed to use the following scheme:

$$y_r = g_1 \nu_1 \oplus g_2 \nu_2 \oplus ..., \tag{13}$$

where $g_1, g_2, ...$ is the Gray code representation of r defined by

$$G(r) = r \oplus \lfloor r/2 \rfloor, \tag{14}$$

i	m_i	$m_i = 4m_{i-2} \oplus 8m_{i-3} \oplus m_{i-3}$	Binary sum	Decimal representation
1	1			
2	3			
3	7			
4	5	$12 \oplus 8 \oplus 1$	$1100 \oplus 1000 \oplus 0001 = 0101$	5
5	7	$28 \oplus 24 \oplus 3$	$11100 \oplus 11000 \oplus 00011 = 00111$	7

Table 2: Generation of the m_i values for Sobol' sequences (the first three values are given)

$\lfloor r/2 \rfloor$ being the largest integer smaller than or equal to x. Combining (13) and (14), the rth term of the Sobol' sequence can be obtained as

$$y_r = y_{r-1} \oplus \nu_c, \tag{15}$$

where ν_c is the ν_i number associated with the rightmost zero in the binary representation of $r-1$. If no zeroes appear, a leading zero must be added.

As an example consider the primitive polynomial

$$P(z) = z^3 + z + 1, \tag{16}$$

and initial values $m_1 = 1$, $m_2 = 3$ and $m_3 = 7$. The corresponding recurrence is $m_i = 4m_{i-2} \oplus 8m_{i-3} \oplus m_{i-3}$. Table 2 shows how to obtain the values m_4 and m_5. To obtain the ν_i, the m_i must first be written in binary form and then the position of the fractional point shifted I positions to the left. Table 3 shows how to do this.

The first five Sobol' points are obtained as follows:

Let $y_0 = 0$ (initial value), using (15) we have

$y_1 = y_0 \oplus \nu 1 = 0.0 \oplus 0.1 = 0.1 = 1 \times 2^{-1} = 0.500$,
$y_2 = y_1 \oplus \nu 2 = 0.1 \oplus 0.11 = 0.01 = 0 \times 2^{-1} + 1 \times 2^{-2} = 0.250$,
$y_3 = y_2 \oplus \nu 1 = 0.01 \oplus 0.10 = 0.11 = 1 \times 2^{-1} + 1 \times 2^{-2} = 0.750$,
$y_4 = y_3 \oplus \nu 3 = 0.11 \oplus 0.111 = 0.001 = 0 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} = 0.125$,
$y_5 = y_4 \oplus \nu 2 = 0.001 \oplus 0.11 = 0.111 = 1 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} = 0.875$.

In the first line we added ν_1 because the binary representation of $r-1=0$ is 0, and the rightmost 0 is in the first position. In the second, we added ν_2 because 1 is 1 in binary representation, so we have to add a leading 0 which is in the second position. In the third we added ν_1 because 2 in binary is 10 and then the rightmost 0 is in the first position, and so on [3].

5. Simulation

An algorithm is applied to find the stationary distribution of ergodic fuzzy Markov chains. Simulations per each size of P are presented based on the

i	m_i	Binary representation	ν_i
1	1	1	0.1
2	3	11	0.11
3	7	111	0.111
4	5	101	0.0101
5	7	111	0.00111

Table 3: Generation of the ν_i values for Sobol' sequences

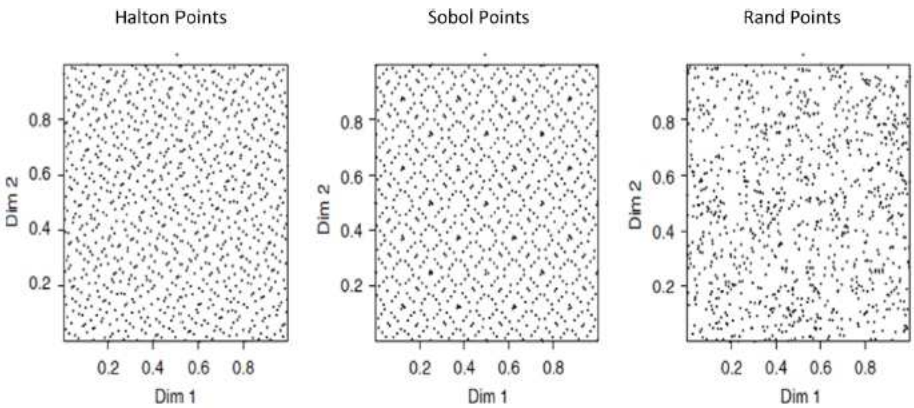

Figure 1

$\vee$	2	3	4	5	6	-	-	-	-	-	-	-	-	-	-
$(m=5)\ NE$	24	387	351	231	7	-	-	-	-	-	-	-	-	-	-
$\vee$	3	4	5	6	7	8	9	-	-	-	-	-	-	-	-
$(m=10)\ NE$	63	313	309	200	74	30	11	-	-	-	-	-	-	-	-
$\vee$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$(m=50)\ NE$	6	58	115	143	165	138	122	80	62	41	27	21	7	6	2
$\vee$	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$(m=100)\ NE$	11	14	35	60	67	82	97	109	94	89	88	75	67	49	24

Table 4: Number of iterations $\vee$ and ergodicity using rand function

results given in Section 3. All elements $\{p_{ij}\}_{i,j=1}^{m}$ of the matrix P are obtained by using the rand function of MATLAB (Table 4), Halton sequences (Table 5) and Sobol' sequences (Table 6).

Number of iterations to obtain $\{\breve{x}_j\}$ is the amount of iterations needed to obtain the greatest eigen fuzzy set $\{\breve{x}_j\}$ according to the method III proposed by Sanchez.

$\vee$	2	3	4	-	-	-	-	-	-	-	-	-	-
$(m=5)\ NE$	336	431	233	-	-	-	-	-	-	-	-	-	-
$\vee$	3	4	5	6	7	-	-	-	-	-	-	-	-
$(m=10)\ NE$	401	324	188	53	34	-	-	-	-	-	-	-	-
$\vee$	5	6	7	8	9	10	11	12	-	-	-	-	-
$(m=50)\ NE$	264	93	102	134	86	147	99	75	-	-	-	-	-
$\vee$	6	7	8	9	10	11	12	13	14	15	16	17	18
$(m=100)\ NE$	198	111	201	89	134	67	73	32	25	41	19	7	3

Table 5: Number of iterations $\vee$ and ergodicity using Halton sequences

$\vee$	2	3	-	-	-	-	-	-	-	-	-	-	-
$(m=5)\ NE$	443	557	-	-	-	-	-	-	-	-	-	-	-
$\vee$	2	3	4	5	-	-	-	-	-	-	-	-	-
$(m=10)\ NE$	367	298	196	139	-	-	-	-	-	-	-	-	-
$\vee$	3	4	5	6	7	8	-	-	-	-	-	-	-
$(m=50)\ NE$	295	171	87	147	168	132	-	-	-	-	-	-	-
$\vee$	5	6	7	8	9	10	11	12	-	-	-	-	-
$(m=100)\ NE$	203	191	112	76	101	265	52	-	-	-	-	-	-

Table 6: Number of iterations $\vee$ and ergodicity using Sobol' sequences

In Tables 4,5 and 6, number of ergodicity and matrix dimension are indicated by NE and m, respectively. From Tables 5 and 6 two results are drawn. First, it is observed that all chains achieve their greatest eigen fuzzy set around a value smaller than m iterations, for instance, if P is 50×50 then most parts of the chains achieve to $\breve{x}_j$ in less than 50 iterations. Second, Sobol' sequences have better efficiency than Halton sequences and rand function. In comparison to the Halton sequences and the rand function, Sobol' sequences provide us stationary distributions using fewer number of iterations $\vee$.

6. Conclusion

The Fuzzy Markov chains are either ergodic or periodic. Though most of the fuzzy Markov chains are periodic, if a fuzzy Markov chain has an ergodic behavior, the greatest eigen fuzzy set $\breve{x}_j$ is used to be its stationary distribution. We may use three numerical approaches, namely rand function of MATLAB, Halton sequences and Sobol' sequences, to generate fuzzy matrix entries of a fuzzy

Markov chain. We revealed that simulation using Sobol' sequences implies better performance than others, it means that to find the stationary distribution of a specific ergodic fuzzy Markov chain is needed less number of iterations.

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