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STRUCTURE OF IDEALS IN SOFT RINGS

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Abstract: It was Molodtsov [3] who introduced the concept of soft set theory, that can be used as a generic mathematical tool for dealing with uncertainty. In this paper we introduce the concepts of soft ideals, prime soft ideals and maximal soft ideals in soft rings and compare soft maximal ideals and soft prime ideals.

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1. Introduction

Researchers in economics, engineering, environmental science, sociology, medical science, and many other fields deal daily with the complexities of modeling uncertain data. The classical methods are not always successful, because the uncertainties appearing in these domains may be of various types. While the probability theory, fuzzy sets, rough sets, and other mathematical tools are well-known and often useful approaches for describing the uncertainty, each of these theories has its inherent difficulties as pointed out by Molodtsov [3]. Consequently, he proposed a completely new approach for modeling vagueness and uncertainty. The so-called soft set theory is free from the difficulties affecting the existing methods. A soft set is a parameterized family of subsets of the universal set. In this article the basic concepts of soft ideals in soft ring theory

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have been demonstrated. Soft ideals, prime soft ideals and maximal soft ideals, are extended to soft rings. We will recall some basic definitions for soft sets introduced by Molodtsov. Throughout this paper U refers to an initial universe set, E is a set of parameters, $P(U)$ is the power set of U, $A \subseteq E$ and $\mathbb{Z}$ is the ring of integer numbers. Molodtsov [3] defined the soft set in the following way:

Definition 1.1. A pair (F, A) is called a *soft set* over U, when F is a mapping given by $F : A \to P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\epsilon \in A, F(\epsilon)$ may be considered as the set of ϵ-approximate elements of the soft set (F, A). Clearly, a soft set is not a set. For illustration, in [3] Molodtsov considered several examples.

Definition 1.2. Assume that (F, A) and (H, B) are two soft sets over a common universe U. We say that (F, A) is a soft subset of (H, B), denoted by $(F, A)\widetilde{\subset}(H, B)$, if it satisfies the following:

(1) $A \subset B$.

(2) $F(x)$ and $H(x)$ are identical approximations for all $x \in A$.

Definition 1.3. Let (F, A) and (H, B) be two soft sets over a common universe U. The bi-intersection of (F, A) and (H, B) is defined as the soft set (K, C) satisfying the following conditions:

(1) $C = A \cap B$.

(2) $K(x) = F(x) \cap H(x)$, for all $x \in C$.

In this case, we write $(K, C) = (F, A)\widetilde{\sqcap}(H, B)$.

Definition 1.4. Assume that (F, A) and (H, B) are two soft sets over a common universe U. The union of (F, A) and (H, B) is defined as the soft set (K, C) satisfying the following conditions:

(1) $C = A \cup B$.

(2) For every $x \in C$

$$K(x) = \begin{cases} F(x) & \text{if } \ x \in A - B, \\ H(x) & \text{if } \ x \in B - A, F(x) \cup H(x) \ \text{ if } \ x \in A \cap B. \end{cases}$$

In this case, we write $(K, C) = (F, A)\widetilde{\cup}(H, B)$.

Definition 1.5. Assume that (F, A) is a soft set over a common universe U. The set of all $x \in A$, where $F(x) \neq \emptyset$ is called the support of the soft set (F, A), i.e. $\mathrm{Supp}(F, A) = \{x \in A | F(x) \neq \emptyset\}$. Furthermore, a soft set (F, A) is said to be non-null set if its support is not equal to the empty set.

Now, we are going to state the definitions of soft ring and soft ideal which is introduced in [4]. From now on, R is a commutative ring and all soft sets are considered over R.

Definition 1.6. Let (F, A) be a non-null soft set over R. Then (F, A) is said a soft ring over R if $F(x)$ is a subring of R, for all $x \in A$.

Example 1.7. Let (F, A) be a soft set over $R = \mathbb{Z}[x]$, where $A = \mathbb{Z}$ and $F : A \to P(\mathbb{Z}[x])$ is set-valued function defined by

$$F(t) = \{f(x) \in \mathbb{Z}[x] | f_0 = tq, q \in \mathbb{Z}$$
$$\text{and } f_0 \text{ is the constant term of the polynomial } f\}.$$

Then $F(t)$ is a subring of $\mathbb{Z}[x]$ for all $t \in A$. Hence, (F, A) is a soft ring over $R = \mathbb{Z}[x]$.

Definition 1.8. Assume that (F, A) is a soft ring over R. A non-null soft set (φ, I) over R is called soft ideal of (F, A), which will be denoted by $(\varphi, I)\widetilde{\triangleleft}(F, A)$, if it satisfies the following conditions:

(1) $I \subset A$.

(2) $\varphi(x)$ is an ideal of $F(x)$ for all x in $Supp(\varphi, I)$.

Example 1.9. Let $A = R = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ and $I = \{0, 2, 4\}$. Consider the set-valued function $F : A \to P(R)$ defined via $F(x) = \{y \in R \mid xy \in \{0, 2, 4\}\}$. Therefore $F(0) = F(2) = F(4) = \mathbb{Z}_6$ and $F(1) = F(3) = F(5) = \{0, 2, 4\}$. As we see, all sets are subrings of R and so (F, A) is a soft ring over R. Furthermore, consider the function $\varphi : I \to P(R)$ given by $\varphi(x) = \{y \in R \mid x + y \in \{0, 2, 4\}\}$. It is clear that, $\varphi(0) = \{0, 2, 4\} \triangleleft F(0) = \mathbb{Z}_6, \varphi(2) = \{0, 2, 4\} \triangleleft F(2) = \mathbb{Z}_6$ and $\varphi(4) = \{0, 2, 4\} \triangleleft F(4) = \mathbb{Z}_6$. Hence, (φ, I) is a soft ideal of (F, A).

2. The Prime and Maximal Soft Ideal

In the classical algebra, the prime ideals and maximal ideals play important rule in the structure of a ring. In this section we introduce the notions of prime soft ideals and soft maximal ideals of a soft ring.

Definition 2.1. Assume that (F, A) is a soft ring over R. A soft ideal (φ, I) of (F, A) is called a prime soft ideal of (F, A), if $\varphi(x)$ is a prime ideal of $F(x)$ for all x *in Supp* (φ, I).

Example 2.2. Let $A = R = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ and $I = \{0\}$. Suppose that $F : A \to P(R)$ is the set-valued function defined via $F(x) = \{y \in R \mid x.y \in \{0, 2\}\}$. Since $F(0) = F(2) = \mathbb{Z}_4$ and $F(1) = F(3) = \{0, 2\}$, (F, A) is a soft ring over R by Definition 1.6. Furthermore, by considering the function $\varphi : I \to P(R)$ given by $\varphi(x) = \{y \in \mathbb{Z}_4 \mid x + 2y = 0\}$, one has $\varphi(0) = \{0, 2\}$ is a prime ideal of $F(0) = \mathbb{Z}_4$ and so (φ, I) is a prime soft ideal of soft ring (F, A).

Theorem 2.3. *Let (F, A) be a soft ring over R and (φ, I) and (θ, J) be prime soft ideals of (F, A) over R. If I and J are two disjoint sets, then $(\varphi, I)\widetilde{\cup}(\theta, J)$ is a prime soft ideal of (F, A).*

Proof. By using Definition 1.4 one can consider $(H, K) = (\varphi, I)\widetilde{\cup}(\theta, J)$, where $K = I \cup J$ and for all $x \in K$,

$$H(x) = \begin{cases} \varphi(x) & \text{if } \ x \in I-, \\ \theta(x) & \text{if } \ x \in J - I. \end{cases}$$

Since $(\varphi, I)\widetilde{\lhd}(F, A)$ and $(\theta, J)\widetilde{\lhd}(F, A)$, then $K \subset A$. Assume that $x \in$ *Supp* (H, K). If $x \in I - J$, then $H(x) = \varphi(x) \neq \emptyset$ is a prime ideal of $F(x)$, since (φ, I) is a prime soft ideal of (F, A). Similarly, if $x \in J - I$, then $H(x) = \theta(x) \neq \emptyset$ is a prime ideal of $F(x)$. Hence, $H(x)$ is a prime ideal of $F(x)$ for all $x \in$ *Supp* (H, K). Therefore, (H, K) is a prime soft ideal of (F, A) by Definition 2.1. □

In the classical algebra, for two disjoint prime ideal P_1 and P_2 of R, one has $P_1 \cap P_2$ is not a prime ideal of R in general. In the following example we will show that the same fact holds for soft rings.

Example 2.4. Let $A = R = \mathbb{Z}$ and $I = J = \{2, 3\}$. The set-valued function $F : A \to P(R)$ defined via $F(x) = \{nx|n \in \mathbb{Z}\} = x\mathbb{Z}$, gives the soft ring (F, A) over R. Consider the set-valued functions $\varphi : I \to P(R)$ and $\theta : J \to P(R)$ given by $\varphi(x) = \{2nx|n \in \mathbb{Z}\} = 2x\mathbb{Z}$ and $\theta(x) = \{3nx|n \in \mathbb{Z}\} = 3x\mathbb{Z}$ respectively. One can easily check that, (φ, I) and (θ, J) are prime soft ideals of soft ring (F, A). Assume that $(H, K) = (\varphi, I)\widetilde{\sqcap}(\theta, J)$, where $K = I \cap J$ and $H(x) = \varphi(x) \cap \theta(x)$ for all $x \in K$. Then $H(2) = \varphi(2)\cap\theta(2) = 4\mathbb{Z} \cap 6\mathbb{Z} = 12\mathbb{Z}$ is not a prime ideal of $F(2)$ and $H(3) = \varphi(3)\cap\theta(3) = 6\mathbb{Z} \cap 9\mathbb{Z} = 18\mathbb{Z}$ is not a prime ideal of $F(3)$. Therefore, $(\varphi, I)\widetilde{\sqcap}(\theta, J)$ is not a prime ideal of (F, A).

Definition 2.5. Let (F, A) be a soft ring over R. A soft ideal (φ, I) of (F, A) is called a maximal soft ideal of (F, A), if $\varphi(x)$ is a maximal ideal of $F(x)$ for all x in *Supp* (φ, I).

Example 2.6. In Example 2.2, (φ, I) is a maximal soft ideal of (F, A), since $\varphi(x)$ is a maximal ideal of $F(x)$ for all x in *Supp* (φ, I).

Since a prime ideal of a commutative ring R may be not a maximal ideal, then every prime soft ideal is not a soft maximal ideal as we show in the following example.

Example 2.7. Let $A = R = \mathbb{Z}$ and $I = \{1\}$. By considering the set-valued functions $F : A \to P(R)$ given by $F(x) = \{nx|n \in \mathbb{Z}\}$ and $\varphi : I \to P(R)$ given by $\varphi(x) = \{n(x-1)|n \in \mathbb{Z}\}$, one has $F(x) = x\mathbb{Z}$ is a subring of R for all $x \in A$ and $\varphi(1) = \{0\}$ is a prime ideal of $F(1) = \mathbb{Z}$. Therefore (F, A) is a soft ring over R and (φ, I) is its prime soft ideal, but (φ, I) is not a maximal soft ideal.

3. Conclusion

The soft sets are deeply related to the fuzzy sets and rough sets. We applied soft sets in ring theory. Hence, by focusing on ideals in soft rings, we have discussed some algebraic properties of soft sets in ring theory and introduced the notions of prime soft ideals and maximal soft ideals, and have given several examples. To extend this work, one could study on other soft ideals such as soft primary ideals in soft rings.

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