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PERIODIC MATRICES AND GABOR FRAMES

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Abstract: A symmetric infinite dimensional matrix $Q = \{ q_{i,j} \}$ is said to be periodic if there exists a real sequence $\{ q_0, q_1, \cdots \}$ such that $q_{i,j} = q_k$ whenever $|i-j| = k$. Determining whether a periodic matrix Q is strongly positive definite is a hard question in general. The goal of this paper is to provide possible answers to this question using frame techniques and the Wiener algebra. In addition, we provide a simple method to construct a Gabor frame sequence whose window function is a step function.

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1. Introduction

A bi-infinite positive matrix Q is *strongly positive* if there exists a constant $c > 0$ such that for any $\mathbf{x} \in \ell_2(\mathbb{Z})$,

$$\mathbf{x}^* Q\, \mathbf{x} \geq c \, \|\mathbf{x}\|^2_{\ell_2(\mathbb{Z})} .$$

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there exists a real sequence $\{ q_0, q_1, \cdots \}$ such that $q_{i,j} = q_k$ whenever $|i-j| = k$. Determining whether a periodic matrix Q is strongly positive definite is a hard question in general. When the real sequence is finite, the question of whether a periodic matrix Q is strongly positive definite is solved in [3] using a Gabor frame. The goal of this paper is to provide possible answers to this question in the more general case using frame techniques and the Wiener algebra. In addition, we provide a simple method to construct a Gabor frame sequence whose window function is a step function. We now define the various notions that were mentioned above.

Suppose H is a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We say a set

$$\{f_k : k \in \mathbb{Z}\}$$

is a *(fundamental) frame* for H if there exist positive numbers A and B such that for each $f \in H$,

$$A \|f\|_H^2 \leq \sum_{k \in \mathbb{Z}} |\langle f, f_k \rangle|^2 \leq B \|f\|_H^2 . \tag{1}$$

The sharpest possible choices for A and B are called the lower frame bound and the upper frame bound respectively. A frame is said to be *tight*, if the upper frame bound is equal to the lower frame bound. The Parseval identity in H shows that every orthonormal basis is a tight frame with frame bounds equal to one. If a set $\{f_k : k \in \mathbb{Z}\}$ satisfies the frame condition (1) for a subspace $\overline{span}\{ f_k : k \in \mathbb{Z} \}$, then we say that $\{f_k : k \in \mathbb{Z}\}$ is a *frame sequence*. In this paper, we focus on Gabor frame sequences in the separable Hilbert space $L_2(\mathbb{R})$. A Gabor system is obtained by applying discrete translations and modulations to a *window function* g defined on $L_2(\mathbb{R})$. Explicitly, a Gabor frame sequence generated by $g \in L_2(\mathbb{R})$ has the form

$$\left\{ e^{im\cdot} g(\cdot - 2\pi n) \, : \, n, m \in \mathbb{Z} \right\}.$$

One of the difficult problems concerning Gabor systems is how to check that a given system is a frame for $L_2(\mathbb{R})$. In general this is still an open problem. Sufficient conditions were established by Daubechies [2], and later, Ron and Shen [7] characterized Gabor frames for $L_2(\mathbb{R})$ using dual Gramian matrices. In their characterization they applied the techniques of dual Gramian matrices to shift-invariant frames for $L_2(\mathbb{R})$, which can be viewed as a Fourier transform of Gabor frames for $L_2(\mathbb{R})$. We adopt their method in this paper, and one of the key ideas of their work will be described in greater detail in next section.

Next, we define

$$W := \left\{ L_a : S^1 \to \mathbb{C} \, : \, L_a(z) = \sum_{n\in\mathbb{Z}} a_n z^n, \quad a := \{ a_n \}_{n\in\mathbb{Z}} \in \ell_1(\mathbb{Z}) \right\},$$

$$\|L_a\|_W := \|a\|_{\ell_1(\mathbb{Z})} \, ,$$

where S^1 is the unit circle in $\mathbb{C}$. We call L_a as the Laurent series corresponding a. We note that

$$a_n = \frac{1}{2\pi} \int_I L_a(e^{i\theta}) e^{-in\theta} \, d\theta, \quad I = [0, 2\pi).$$

The set W with the usual pointwise operations is a commutative Banach algebra, which is called the Wiener algebra. Let $GW \subset W$ be the subgroup of the invertible elements of W. Then the classical Wiener's lemma states that if a periodic function f has an absolutely convergent Fourier series and never vanishes, then $1/f$ has an absolutely convergent Fourier series:

Theorem 1. (see [1], p. 6)

$$GW = \left\{ L_a \in W \, : \, L_a(z) \neq 0, \quad \forall z \in S^1 \right\}.$$

An equivalent formulation that is more suitable for this paper considers the following Toeplitz matrix generated by $a \in \ell_1(\mathbb{Z})$:

$$T_a := \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & a_0 & a_{-1} & a_{-2} & \\ & a_1 & \underline{a_0} & a_{-1} & \\ & a_2 & a_1 & a_0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}.$$

Here $\underline{a_0}$ denotes the entry a_0 located in the zero row and zero column position. We consider this bi-infinite matrix as the convolution operator $T_a b = a * b$ acting on $\ell_2(\mathbb{Z})$. Using Young's inequality we have the following proposition.

Proposition 2. *If $a \in \ell_1(\mathbb{Z})$, then $T_a : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ is a bounded operator and*

$$\|T_a\| \leq \|a\|_{\ell_1(Z)} \, .$$

In this case, Wiener's lemma gives the following theorem.

Theorem 3. (see [5]) *If $a \in \ell_1(\mathbb{Z})$ and T_a is invertible as an operator on $\ell_2(\mathbb{Z})$, then L_a does not have zeros on S^1.*

For the converse direction we note that if $L_a \in GW$, then there exists $L_b(z)$ such that $T_a^{-1} = T_b$ and $L_b(z) = \dfrac{1}{L_a(z)}$, $z \in S^1$.

2. Preliminaries

Let $g \in L_2(\mathbb{R})$. We denote the modulation of g by an integer m as

$$M_m g(t) := e^{imt} g(t), \quad t \in \mathbb{R}.$$

If $G := \{g_n : n \in \mathbb{Z}\}$ is a system of functions in $L_2(\mathbb{R})$, then the modulation system generated by G is the system

$$\{\, M_m g_n \; : \; m, n \in \mathbb{Z}\,\}.$$

A modulation system that is also a frame for a subspace of $L_2(\mathbb{R})$ will be called a *modulation frame sequence* for $L_2(\mathbb{R})$. We formally consider the following frame operator [6]:

$$Sf := \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \langle f, M_m g_n \rangle \, M_m g_n. \tag{2}$$

We now derive a matrix formulation of (2) when f is any compactly supported function in $L_2(\mathbb{R})$. Let $I := [0, 2\pi)$. Then we have

$$\begin{aligned} Sf &= \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \left[\int_{\mathbb{R}} f(t)\overline{g_n}(t) e^{-imt}\, dt \right] e^{imt} g_n(t) \\ &= 2\pi \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \left[\sum_{j\in\mathbb{Z}} \frac{1}{2\pi} \int_{I+2\pi j} f(t)\overline{g_n}(t) e^{-imt}\, dt \right] e^{imt} g_n(t). \end{aligned}$$

Substitution yields that for f compactly supported,

$$\begin{aligned} Sf &= 2\pi \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \left[\sum_{j\in\mathbb{Z}} \frac{1}{2\pi} \int_I f(t+2\pi j)\overline{g_n}(t+2\pi j) e^{-imt}\, dt \right] e^{imt} g_n(t) \\ &= 2\pi \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \left[\frac{1}{2\pi} \int_I [f, g_n](t) e^{-imt}\, dt \right] e^{imt} g_n(t), \end{aligned}$$

where
$$[f,g](t) := \sum_{j\in\mathbb{Z}} f(t+2\pi j)\overline{g}(t+2\pi j).$$

Since $[f,g_n] \in L_2(I)$ and $\{\, e^{im\cdot}\,\}_{m\in\mathbb{Z}}$ is an orthogonal basis of $L_2(I)$,
$$\sum_{m\in\mathbb{Z}} \left[\frac{1}{2\pi}\int_I [f,g_n](t)e^{-imt}\,dt\right] e^{imt} = [f,g_n](t).$$

Consequently, for any compactly supported $f \in L_2(\mathbb{R})$, we have
$$Sf = 2\pi \sum_{n\in\mathbb{Z}} [f,g_n]g_n, \tag{3}$$

which implies that
$$\begin{aligned} Sf(t) &= 2\pi \sum_{n\in\mathbb{Z}} [f,g_n](t)g_n(t) \\ &= 2\pi \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} f(t+2\pi k)\overline{g_n}(t+2\pi k)g_n(t), \\ &= 2\pi \sum_{k\in\mathbb{Z}} \left[\sum_{n\in\mathbb{Z}} g_n(t)\overline{g_n}(t+2\pi k)\right] f(t+2\pi k), \end{aligned}$$

where interchanging the order of summation is valid because f has compact support. Since $[f,g_n]$ is 2π-periodic, we have that
$$Sf(t+2\pi j) = 2\pi \sum_{k\in\mathbb{Z}} \left[\sum_{n\in\mathbb{Z}} g_n(t+2\pi j)\overline{g_n}(t+2\pi k)\right] f(t+2\pi k).$$

As an immediate consequence of the above we get
$$\begin{bmatrix} \vdots \\ Sf(t-2\pi) \\ Sf(t) \\ Sf(t+2\pi) \\ \vdots \end{bmatrix} = G(t)\cdot \begin{bmatrix} \vdots \\ f(t-2\pi) \\ f(t) \\ f(t+2\pi) \\ \vdots \end{bmatrix}, \tag{4}$$

where
$$G(t) := \left[2\pi \sum_{n\in\mathbb{Z}} g_n(t+2\pi j)\overline{g_n}(t+2\pi k)\right]_{(j,k)\in\mathbb{Z}^2}, \quad t \in \mathbb{R},$$

is called the dual Gramian matrix [7].

In this paper we consider a function of the form $g := \sum_{n\in\mathbb{Z}} b_k \chi_{I+2\pi k}, \quad I := [0, 2\pi)$ and define $g_n(t) := g(t - 2\pi n)$. The (j, k)-th entry of the corresponding matrix is then

$$G(t)(j,k) = 2\pi \sum_{n\in\mathbb{Z}} g(t - 2\pi n + 2\pi j)\bar{g}(t - 2\pi n + 2\pi k)$$

$$= 2\pi \sum_{n\in\mathbb{Z}} b_{-n+j} b_{-n+k} = 2\pi \sum_{m\in\mathbb{Z}} b_{-m} b_{k-j-m} = 2\pi(\widetilde{b} * b)_{k-j},$$

where $\widetilde{b}_n = b_{-n}$. Since $G(t)(j,k) = G(t)(k,j)$, we have $G(t)(j,k) = 2\pi(\widetilde{b} * b)_{|k-j|}$. We define the sequence $G := \{ G_n \}_{n\in\mathbb{Z}}$ corresponding to g as

$$G_n := 2\pi(\widetilde{b} * b)_n. \tag{5}$$

Then for any $t \in \mathbb{R}$, we obtain the following periodic matrix:

$$G(t) = G(0) =: \mathbb{G} =: \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & G_0 & G_1 & G_2 & \\ & G_1 & \underline{G_0} & G_1 & \\ & G_2 & G_1 & G_0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}. \tag{6}$$

Then the question in the introduction leads us to the following objective.

Objective 4. *Determine when $\mathbb{G}$ is strongly positive and find a correspondence between $\mathbb{G}$ and a Wiener element.*

A special case of the objective has the following answer: if $\mathbb{G}$ is invertible, then $\mathbb{G}$ is strongly positive since

$$\mathbf{x}^* \mathbb{G}\, \mathbf{x} \geq \frac{\|\mathbf{x}\|^2_{\ell_2(\mathbb{Z})}}{\|\mathbb{G}^{-1}\|}. \tag{7}$$

Thus, to answer Objective 4, we study when $\mathbb{G}$ is invertible. We also note that $\frac{1}{\|\mathbb{G}^{-1}\|}$ is the lower frame bound when g generates a modulation frame sequence. One can consider the matrix $\mathbb{G}$ as the Toeplitz matrix generated by $G := \{ G_n \}_{n\in\mathbb{Z}}$, where $G_k = G_{-k}$ if $G \in \ell_1(\mathbb{Z})$. We reformulate in this way since the invertibility of a Toeplitz matrix generated by $L_a \in W$ is well studied by Wiener [1, 5]. This consideration leads us to study the relationship between a frame and a Wiener element.

3. Frames and Wiener Elements

As one can see from previous sections, the key connection between frames and elements of the Wiener algebra is the matrix representation. The notion of a frame is characterized using matrix representation in [7], which is restated here:

Proposition 5. *Let $g := \sum_{n\in\mathbb{Z}} b_k\chi_{I+2\pi k}$ be a function in $L_2(\mathbb{R})$. Then we have the following:*

(I) The frame operator S is bounded if and only if $\|\mathbb{G}\| < \infty$.

(II) g generates a Gabor frame sequence if and only if

$$\|\mathbb{G}\| < \infty \text{ and } \left\|\mathbb{G}^{-1}\right\| < \infty.$$

Thus, if $g := \sum_{n\in\mathbb{Z}} b_k\chi_{I+2\pi k}$ generates a frame sequence for $L_2(\mathbb{R})$, then the corresponding periodic matrix $\mathbb{G}$ is strongly positive. The following theorem shows the first relationship between the notion of a frame and a Wiener element.

Theorem 6. *Let $g := \sum_{n\in\mathbb{Z}} b_k\chi_{I+2\pi k}$ be a function in $L_2(\mathbb{R})$. If the corresponding sequence G defined in (5) is absolutely convergent and if g generates a frame sequence, then the Wiener element L_G has no zeros on the unit circle.*

Proof. Since g generates a frame sequence, by Proposition 5, the corresponding Toeplitz matrix $\mathbb{G}$ is invertible. Thus by Theorem 3, L_G has no zeros on the unit circle. ☐

From Young's Inequality, we have $\sum_{n\in\mathbb{Z}} |G_n| = \sum_{n\in\mathbb{Z}} \left|\widetilde{b} * b(n)\right| \leq \|b\|^2_{\ell_2(\mathbb{Z})}$. Thus we have the following immediate consequence.

Corollary 7. *Let $g := \sum_{n\in\mathbb{Z}} b_k\chi_{I+2\pi k}$ be a function in $L_2(\mathbb{R})$. If $b \in \ell_1(\mathbb{Z})$ and g generates a frame sequence, then the Wiener element L_G has no zeros on the unit circle.*

The previous theorem implies that if a step function g generates a frame, then the corresponding periodic matrix $\mathbb{G}$ is a Toeplitz matrix generated by a

Wiener element. The following objective concerns the opposite implication.

Objective 8. *For a given $a \in \ell_1(\mathbb{Z})$ such that $L_a \in GW$ and T_a is symmetric we want to construct a function g which generates a frame sequence corresponding to a.*

For this objective, we observe the following matrix factorization of $\mathbb{G}$:

$$\mathbb{G} = 2\pi \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & b_0 & b_{-1} & b_{-2} & \\ & b_1 & \underline{b_0} & b_{-1} & \\ & b_2 & b_1 & b_0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \cdot \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & b_0 & b_1 & b_2 & \\ & b_{-1} & \underline{b_0} & b_1 & \\ & b_{-2} & b_{-1} & b_0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}. \tag{8}$$

From this factorization, we have the following equivalent statement of Objective 8: For a given $a \in \ell_1(\mathbb{Z})$ such that $L_a \in GW$ and T_a is symmetric, when can we have the following factorization for an infinite matrix T_a?

$$T_a = LL^*.$$

The Cholesky decomposition gives us one possible answer.

Proposition 9. *If $\mathbb{G}$ is a bounded operator, then $\mathbb{G}$ is positive.*

Theorem 10. *For a given $a \in \ell_1(\mathbb{Z})$ such that $L_a \in GW$ and T_a is symmetric and positive, there exists a function g which generates a frame sequence and whose corresponding periodic matrix becomes T_a.*

Proof. Using the Cholesky decomposition, we have $T_a = LL^*$, where L is a lower triangular matrix with strictly positive diagonal entries. Then $g := \sum_{n\in\mathbb{Z}} L(0,n)\chi_{I+2\pi n}$ is the desired function. $\square$

We note that the Cholesky decomposition for infinite matricies in general is not easy to perform. However, if L_a is a trigonometric polynomial, i.e., a is finitely supported, and takes nonnegative values on the unit circle, then by the Fejér-Riesz factorization, L_a can be factorized as

$$L_a = \overline{L_b} L_b, \quad L_b = b_0 + b_1 z + \cdots + b_n z^n, \quad |z| = 1,$$

for some $n \in \mathbb{N}$. That is, finding $L_a = \overline{L_b} L_b$ is equivalent to finding a lower Toeplitz matrix L such that $T_a = LL^*$. Thus in this case, a construction of the corresponding function g which generates a frame sequence is easier. We also obtain the following result which is the converse of Theorem 6.

Corollary 11. *Let $g := \sum_{n\in\mathbb{Z}} b_k \chi_{I+2\pi k}$ be a function in $L_2(\mathbb{R})$. If the corresponding Laurent series L_G is a Wiener element which has no zeros on the unit circle, then g generates a frame sequence.*

Proof. By Theorem 3, the corresponding matrix $\mathbb{G}$ of g is invertible. Thus g generates a frame sequence by Proposition 5. □

If a Wiener element has no zeros on the unit circle and is continuous, then norm of the corresponding Toeplitz matrix as an operator acting on $\ell_2(\mathbb{Z})$ is same as the L_∞-norm of the Wiener element [4]. In this case, the corresponding Toeplitz matrix of the inverse of the Wiener element is the inverse of the Toeplitz matrix. Thus we have the following corollary.

Corollary 12. *Let $g := \sum_{n\in\mathbb{Z}} b_k \chi_{I+2\pi k}$ be a function in $L_2(\mathbb{R})$. If the corresponding Laurent series L_G is a Wiener element which is continuous and has no zeros on the unit circle, then g generates a frame sequence with the frame bounds $\|L_G\|_{L_\infty(I)}$ and $\|1/L_G\|_{L_\infty(I)}$.*

We note that for the construction of such a g, checking the condition that G is absolutely convergent is not easy. By assuming $b \in \ell_1(\mathbb{Z})$, we obtain a frame sequence immediately.

Theorem 13. *Let $b \in \ell_1(\mathbb{Z})$. Suppose L_b has no zeros on the unit circle. Then g generates a frame sequence with the frame bounds $\|\mathbb{G}\|$ and $\left\|\mathbb{G}^{-1}\right\|$.*

Proof. Since $b \in \ell_1(\mathbb{Z}) \subset \ell_2(\mathbb{Z})$, g is in $L_2(\mathbb{R})$. Since $L_b \in GW$, T_b is an invertible operator so is $\mathbb{G} = T_b T_b^*$, which implies that g generates a frame sequence with the frame bounds $\|\mathbb{G}\|$ and $\left\|\mathbb{G}^{-1}\right\|$. □

Theorem 13 says that any Laurent series whose zeros are not on the unit circle with ℓ_1 coefficients generates a frame sequence. For example, consider the following elementary analytic function $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$. Since the zeros of this function are not on the unit circle, the corresponding Toeplitz matrix is an invertible operator. Thus

$$g := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \chi_{I+2n\pi}$$

generates a frame sequence.

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