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THE MAIN DIAGONAL METHOD IN C^1 GLOBAL OPTIMIZATION PROBLEM

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Abstract: In this paper we present a new method for solving a C^1 global optimization problem on a box in R^n. We consider the two vertices of the main diagonal to construct a convex quadratic lower bound function of the objective function which is considered of class C^1. A branch and bound algorithm with exhaustive $w-$*subdivision* is proposed and its convergence is shown.

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1. Introduction

We consider the following global optimization problem:

$$(P)\begin{cases} \min b(s) \\ s \in S \end{cases}$$

with b a C^1 nonconvex function and S is a box in R^n. Even if constraints are simple, the problem remains very difficult to find global minimum of b. The principal existing methods use $(n+1)$ vertices of a box like the arithmetic interval method, or consider the C^2 objective function like the αBB method. In [7] they use $(n+1)$ vertices of the box to construction a right simplex and feasible simplex and by solving a linear system, they find a lower bound of the objective function. In [1] the αBB method is used to compute a global minimum

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of the objective function which consists in a construction of convex lower bound function of the C^2 functions with the rigorous method for the calculation of the α parameters and the branch and bound algorithm is used. The review of principal methods are presented in [3] and [6]. Efficient diagonal partitions are discussed in [9]. In [5] the interval method is used to compute global optimum of C^2 function. In [4] the branch and bound algorithms HOL^1 and HOL^n are presented to compute global minimum of the holer function in an interval of R and in a box of R^n respectively. In our method we use the two vertices of the main diagonal to find a convex quadratic lower bound function. In our branch and bound algorithm, we solve at each iteration a triangular system to compute a lower bound. For branching, we use the exhaustive $w-subdivision$. Our method is an extension of [8]. The paper is organized as follows: In Section 2 we present the main results. The algorithm and its convergence are presented in Section 3. A numerical example found in the literature is treated in Section 4.

2. Main New Results

We propose a new lower bound function by using the two vertices of the main diagonal of a box, s^0 and s^1 such that $\forall s \in S, s_i^0 \leq s_i \leq s_i^1, i = 1, ..., n$ with $s^0 = (s_1^0, ..., s_n^0), s^1 = (s_1^1, ..., s_n^1), s = (s_1, ..., s_n)$

$$LB(s) = \sum_{i=0}^{1}[(b(s^i) + \sum_{j=1}^{n}(s_j - s_j^i)(l\cos i\frac{\pi}{2} + L\sin i\frac{\pi}{2})]w_i(s)$$

with

$$l_i \leq \frac{\partial b(\xi)}{\partial s_i} \leq L_i \; ; i = 1, .., n \quad ; \xi \in S$$

$$l = \min_{i=1,..,n} l_i \qquad \text{and} \qquad L = \max_{i=1,..,n} L_i$$

$$w_0(s) = \frac{\sum_{j=1}^{n}(s_j^1 - s_j)}{\sum_{j=1}^{n}(s_j^1 - s_j^0)} \quad \text{and} \quad w_1(s) = \frac{\sum_{j=1}^{n}(s_j - s_j^0)}{\sum_{j=1}^{n}(s_j^1 - s_j^0)}.$$

One has the properties of $w_i(s)$:

$$0 \leq w_i(s) \leq 1; i = 0, 1$$

$$w_i(s^j) = \delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}, \quad i = 0, 1; \; j = 0, 1,$$

$$\sum_{i=0}^{1} w_i(s) = 1.$$

Theorem 2.1. *The lower bound function coincides with the function b at the vertices of the main diagonal.*

Proof.

$$\begin{aligned}
LB(s^0) =& \sum_{i=0}^{1}[(b(s^i) + \sum_{j=1}^{n}(s_j^0 - s_j^0)(l\cos i\frac{\pi}{2} + L\sin i\frac{\pi}{2})]w_i(s^0) \\
=&[(b(s^0) + \sum_{j=1}^{n}(s_j^0 - s_j^0)(l\cos 0\frac{\pi}{2} + L\sin 0\frac{\pi}{2})]w_0(s^0) \\
& \hspace{6cm} \text{(because } w_1(s^0) = 0\text{)} \\
=&b(s^0), \\
LB(s^1) =& \sum_{i=0}^{1}[(b(s^i) + \sum_{j=1}^{n}(s_j^1 - s_j^i)(l\cos i\frac{\pi}{2} + L\sin i\frac{\pi}{2})]w_i(s^1) \\
=&[(b(s^1) + \sum_{j=1}^{n}(s_j^1 - s_j^1)(l\cos 1\frac{\pi}{2} + L\sin 1\frac{\pi}{2})]w_1(s^1) \\
& \hspace{6cm} \text{(because } w_0(s^1) = 0\text{)} \\
=&b(s^1)
\end{aligned}$$

□

Theorem 2.2.

$$LB(s) \leq b(s); \forall s \in S.$$

Proof. $LB(s)$ and $b(s)$ coincide at the vertices of the main diagonal s^0 et s^1, for the other points of S one has

$$b(s) = b(s^0) + (s - s^0)\nabla b(\xi) \geq b(s^0) + \sum_{j=1}^{n}(s_j - s_j^0)l, \tag{1}$$

or

$$b(s) = b(s^1) + (s - s^1)\nabla b(\eta) \geq b(s^1) + \sum_{j=1}^{n}(s_j - s_j^1)L. \tag{2}$$

On the other hand,

$$\begin{aligned} LB(s) &= \sum_{i=0}^{1}[b(s^i)+\sum_{j=1}^{n}(s_j-s_j^i)(l\cos i\frac{\pi}{2}+L\sin i\frac{\pi}{2})]w_i(s) \\ &\le \max_{i=0,1}[b(s^i)+\sum_{j=1}^{n}(s_j-s_j^i)(l\cos i\frac{\pi}{2}+L\sin i\frac{\pi}{2})]\sum_{i=0}^{1}w_i(s) \\ &\le \max_{i=0,1}[b(s^i)+\sum_{j=1}^{n}(s_j-s_j^i)(l\cos i\frac{\pi}{2}+L\sin i\frac{\pi}{2})] \le b(s) \end{aligned}$$

(i.e. according to (1) and (2)), thus $LB(s) \le b(s)$, $\forall s \in S$. □

Theorem 2.3. *The lower bound function is quadratic.*

Proof. One has

$$\begin{aligned} LB(s) &= \sum_{i=}^{1}[(b(s^i)+\sum_{j=1}^{n}(s_j-s_j^i)(l\cos i\frac{\pi}{2}+L\sin i\frac{\pi}{2})]w_i(s) \\ &= [(b(s^0)+\sum_{j=1}^{n}(s_j-s_j^0)l]w_0(s)+[(b(s^1)+\sum_{j=1}^{n}(s_j-s_j^1)L]w_1(s) \\ &= b(s^0)w_0(s)+b(s^1)w_1(s)+\Delta_d w_0(s).w_1(s)(l-L) \end{aligned}$$

with

$$\Delta_d = \sum_{j=1}^{n}(s_j^1-s_j^0).$$

We compute

$$\begin{aligned} w_0(s).w_1(s) &= \frac{1}{\Delta_d^2}\sum_{j=1}^{n}(s_j^1-s_j).\sum_{j=1}^{n}(s_j-s_j^0) \\ &= \frac{1}{\Delta_d^2}(\sum_{j=1}^{n}s_j^1-\sum_{j=1}^{n}s_j).(\sum_{j=1}^{n}s_j-\sum_{j=1}^{n}s_j^0) \\ &= \frac{1}{\Delta_d^2}[-\left(\sum_{j=1}^{n}s_j\right)^2+\sum_{j=1}^{n}s_j[\sum_{j=1}^{n}s_j^0+\sum_{j=1}^{n}s_j^1]-\sum_{j=1}^{n}s_j^1.\sum_{j=1}^{n}s_j^0]. \end{aligned}$$

The lower bound function is written as a quadratic function

$$\begin{aligned} LB(s) &= b(s^0)w_0(s) + b(s^1)w_1(s) + \Delta_d w_0(s).w_1(s)(l-L) \\ &= b(s^0)w_0(s) + b(s^1)w_1(s) + \\ &\quad + \frac{(l-L)}{\Delta_d}[-\left(\sum_{j=1}^{n} s_j\right)^2 + \sum_{j=1}^{n} s_j[\sum_{j=1}^{n} s_j^0 + \sum_{j=1}^{n} s_j^1] - \sum_{j=1}^{n} s_j^1 . \sum_{j=1}^{n} s_j^0]. \end{aligned}$$

□

Theorem 2.4. *The lower bound function is convex.*

Proof. All the elements of the hessian matrix of the lower bound function are equal to $\frac{(L-l)}{\Delta_d} > 0$, its eigenvalues are positive or zero, then it is positive semi-definite, thus the lower bound function is convex. □

3. Algorithm and its Convergence

We now describe our branch and bound algorithm.

3.1. Algorithm

Initialization. Let $\varepsilon > 0$. Compute by interval method l^0 and L^0.

$$l^0 = \min_{i=1,..,n} l_i \ , \ \ l_i \leq \frac{\partial b(s)}{\partial s_i}, \forall s \in S$$

and

$$L^0 = \max_{i=1,..,n} L_i \ \ \frac{\partial b(s)}{\partial s_i} \leq L_i, \forall s \in S.$$

Iteration 0
Let $T^0 = S$, solve the triangular system

$$\begin{cases} \sum_{i=1}^{n} s_i = \frac{1}{2}\sum_{i=1}^{n}(s_i^0 + s_i^1) + \frac{b(s^0)-b(s^1)}{2(L^0-l^0)} \\ \frac{s_n - s_n^0}{s_n^1 - s_n^0} = \frac{s_{n-1} - s_{n-1}^0}{s_{n-1}^1 - s_{n-1}^0} = \text{.........} = \frac{s_1 - s_1^0}{s_1^1 - s_1^0} \end{cases}$$

to obtain a solution $\overline{s}^0$.
If $\overline{s}^0 \notin S$
project $\overline{s}^0$ on S to obtain s^0 or s^1 which is the global optimal solution and the

algorithm terminates.
else
set

$$UB_0 = \min\{\min_{i=0,1} b(s^i),\ b(\overline{s}^0)\};\ LB_0 = LB(\overline{s}^0).$$

if $UB_0 - LB_0 \le \varepsilon$ then stop $\overline{s}^0$ is $\varepsilon-$optimal solution
else set $M \leftarrow T^0$, $\quad k \leftarrow 1$, and go to iteration k.
Iteration $k = 1, 2, 3, ...$

$k1$. Set

$$T^k = \prod_{i=1}^{n} \left[s_i^{1k}, s_i^{0k}\right] \in M.$$

the box such that $LB_k = LB(T^k)$ and $\overline{s}^k$ the minimum of LB on T^k
$k2$. divide T^k on two boxes T^{k_1}, T^{k_2} by the $w - subdivision$ via $\overline{s}^k$.
$k3$. For $i = 1, 2$.
Compute L^{k_i} et l^{k_i} on T^{k_i} by interval method,
solve the triangular system

$$\begin{cases} \sum_{i=1}^n s_i = \frac{1}{2}\sum_{i=1}^n (s_i^{0k} + s_i^{1k}) + \frac{b(s^{0k}) - b(s^{1k})}{2(L^k - l^k)} \\ \frac{s_n - s_n^{0k}}{s_n^{1k} - s_n^{0k}} = \frac{s_{n-1} - s_{n-1}^{0k}}{s_{n-1}^{1k} - s_{n-1}^{0k}} = ... = \frac{s_1 - s_1^{0k}}{s_1^{1k} - s_1^{0k}} \end{cases}$$

to obtain $\overline{s}^{k_i}$,
if $\overline{s}^{k_i} \notin T^{k_i}$
project $\overline{s}^{k_i}$ on T^{k_i} to obtain one of the vertices of $T^{k_i} s^{0k_i}$ or s^{1k_i}
set $UB_{ki} = LB_{ki} = b(s^{0k_i})$ or $UB_{ki} = LB_{ki} = b(s^{1k_i})$
else continue.

$k4$. Update the upper bound

$$UB_k = \min\{UB_{k-1}, b(s^{0k_1}), b(s^{1k_1}), b(s^{0k_2}), b(s^{1k_2}) b(\overline{s}^{k_1}), b(\overline{s}^{k_2})\}$$

Let s^k the actual best solution i.e $b(s^k) = UB_k$
set

$$M \leftarrow M \cup \{T^{k_i} : LB(T^{k_i}) < UB_k - \varepsilon,\ i = 1, 2\} \backslash \{T^k\}.$$

Update the lower bound

$$LB_k = \min\{LB(T) : T \in M\}$$

$k5$. If $M = \emptyset$ then stop s^k is the optimal solution
else set $k \leftarrow k + 1$ and return to $k1$.

3.2. Convergence

Theorem 3.1. *Our algorithm using the exhaustive $w-$subdivion converges to an optimal solution.*

Proof. It suffices to show that $\lim\limits_{k\to\infty}(UB_k - LB_k) = 0$.

Let

$$\Delta_{dk} = \sum_{j=1}^{n}(s_j^{1k} - s_j^{0k}),$$

$$w_{0k}(s) = \frac{\sum_{j=1}^{n}(s_j^{1k} - s_j)}{\sum_{j=1}^{n}(s_j^{1k} - s_j^{0k})} \quad \text{and} \quad w_{1k}(s) = \frac{\sum_{j=1}^{n}(s_j - s_j^{0k})}{\sum_{j=1}^{n}(s_j^{1k} - s_j^{0k})}.$$

One has

$$\begin{aligned}
0 \quad \leq \quad & UB_k - LB_k = b(s^k) - LB_k = b(s^{0k}) + (s - s^{0k})\nabla b(\xi) \\
& -\sum_{i=0}^{1}[(b(s^{ik}) + \sum_{j=1}^{n}(s_j - s_j^{ik})(l^k \cos i\frac{\pi}{2} + L^k \sin i\frac{\pi}{2})]w_{ik}(s) \\
\leq \quad & b(s^{0k}) + (s - s^{0k})\nabla b(\xi) \\
& -\min_{i=0,1}[(b(s^{ik}) + \sum_{j=1}^{n}(s_j - s_j^{ik})(l^k \cos i\frac{\pi}{2} + L^k \sin i\frac{\pi}{2})]\sum_{i=0}^{1} w_{ik}(s) \\
\leq \quad & b(s^{0k}) + \sum_{j=1}^{n}(s_j - s_j^{0k})L^k - \min\{b(s^{0k}) + \sum_{j=1}^{n}(s_j - s_j^{0k})l^k; b(s^{1k}) \\
& + \sum_{j=1}^{n}(s_j - s_j^{1k})L^k\}.
\end{aligned}$$

We have two cases:

(1)

$$\min\{b(s^{0k}) + \sum_{j=1}^{n}(s_j - s_j^{0k})l^k, b(s^{1k}) + \sum_{j=1}^{n}(s_j - s_j^{1k})L^k\} = b(s^{0k}) + \sum_{j=1}^{n}(s_j - s_j^{0k})l^k,$$

then

$$\begin{aligned} 0 &\leq UB_k - LB_k = b(s^k) - LB_k \leq b(s^{0k}) + \sum_{j=1}^{n}(s_j - s_j^{0k})L^k \\ &\quad - \left(b(s^{0k}) + \sum_{j=1}^{n}(s_j - s_j^{0k})l^k \right) \\ &\leq \sum_{j=1}^{n}(s_j - s_j^{0k})(L^k - l^k) \\ &\leq \Delta_{dk}(L^k - l^k) \to 0 \end{aligned}$$

because $\Delta_{dk} \to 0$ when $k \to \infty$ (i.e we use the exhaustive $w - subdivision$).

(2)

$$\min\{b(s^{0k})+\sum_{j=1}^{n}(s_j-s_j^{0k})l^k, b(s^{1k})+\sum_{j=1}^{n}(s_j-s_j^{1k})L^k\} = b(s^{1k})+\sum_{j=1}^{n}(s_j-s_j^{1k})L^k.$$

Then:

$$\begin{aligned} 0 &\leq UB_k - LB_k = b(s^k) - LB_k \leq b(s^{0k}) + \sum_{j=1}^{n}(s_j - s_j^{0k})L^k \\ &\quad - \left(b(s^{1k}) + \sum_{j=1}^{n}(s_j - s_j^{1k})L^k \right) \\ &= b(s^{0k}) - b(s^{1k}) + L^k(\sum_{j=1}^{n}(s_j - s_j^{0k}) - \sum_{j=1}^{n}(s_j - s_j^{1k})) \\ &= (s^{0k} - s^{1k})\nabla b(\xi) + L^k(\sum_{j=1}^{n}(s_j^{1k} - s_j^{0k}) \\ &\leq (\sum_{j=1}^{n}(s_j^{1k} - s_j^{0k})(-l^k) + L^k(\sum_{j=1}^{n}(s_j^{1k} - s_j^{0k}) \\ &= \sum_{j=1}^{n}(s_j^{1k} - s_j^{0k})(L^k - l^k) \\ &\leq \Delta_{dk}(L^k - l^k) \to 0 \end{aligned}$$

because $\Delta_{dk} \to 0$ when $k \to \infty$ (i.e we use the exhaustive $w - subdivision$). $\square$

Remark. To find the minimum of the lower bound function on T^{k_i} we have to vanish its gradient and we obtain one equation

$$\sum_{i=1}^{n} s_i = \frac{1}{2}\sum_{i=1}^{n}(s_i^{0k_i} + s_i^{1k_i}) + \frac{b(s^{0k_i}) - b(s^{1k_i})}{2(L-l)}.$$

This minimum must be on the main diagonal then we have to add $(n-1)$ equations

$$\frac{s_n - s_n^{0k_i}}{s_n^1 - s_n^{0k_i}} = \frac{s_{n-1} - s_{n-1}^{0k_i}}{s_{n-1}^{1k_i} - s_{n-1}^{0k_i}} = \ldots\ldots\ldots = \frac{s_1 - s_1^{0k_i}}{s_1^1 - s_1^{0k_i}}.$$

Thus we obtain a triangular system with n equations and n unknowns which admits a unique solution.

We have tree cases:

First case: this minimum belongs to the interior of T^{k_i}, then we use the exhaustive $w - subdivision$.

Second case: this minimum is equal to one vertex of the main diagonal, then we have the lower bound which is equal to the upper bound on T^{k_i} and it is deleted for the next iteration.

Third case: this minimum does not belong to T^{k_i} then we project it on T^{k_i}, we obtain one of the vertices of main diagonal and we have the lower bound which is equal to the upper bound on T^{k_i} and it is deleted for the next iteration.

4. Numerical Example

Example from [4]**:** $b(s_1, s_2) = -\sin(s_1)\sin(s_1 s_2)$ on $[0,4]\times[0,4]$.

The solution found by our method is $(1.5588, .99655)$ and $b(1.5588, .99655) = -.99978; LB = -1.1328$, with practically the same precision as in [4], the number of function evaluations is 99 which is significantly lower than that of [4] which is equal to 1013.

5. Conclusion

We have proposed a new lower bound for C^1 global optimization problem by using the main diagonal. Our calculations are all explicit. The convergence of our algorithm is shown and a numerical example found in the literature is treated.

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