International Journal of Applied Mathematics

Volume 25 No. 6 2012, 753-761

REMOTAL SUBSESTS IN $L(H,H)$

R.A.G. AlMirbati[1 §], M.M. Al-Gharabli[2], R. Khalil[3]

[1]Department of Mathematics and Natural Sciences
Gulf University for Science and Technology (GUST)
Hawalli, 32093, KUWAIT
e-mail: mirbati.r@gust.edu.kw

[2]Department of Mathematics
King Fahd University of Petroleum and Minerals (KFUPM)
Dhahran, KINGDOM OF SAUDI ARABIA
e-mail: mahfouz@kfupm.edu.sa

[3]Department of Mathematics
The University of Jordan
Amman, 11942, JORDAN
e-mail: roshdi@ju.edu.jor

Abstract: Let X be a Banach space and Y be a closed bounded subset of X. We call Y remotal in X, if for every $x \in X$ there exists some $y \in Y$ such that

$$\| x - y \| \geq \| x - z \|, \quad \forall\, z \in Y.$$

It is called uniquely remotal in X, if for every $x \in X$ there is a unique $y \in Y$ such that

$$\| x - y \| \geq \| x - z \|, \quad \forall\, z \in Y.$$

In this paper, we present some new results on remotality of sets in the Banach space $L(H,H)$.

AMS Subject Classification: 41A65, 41A28
Key Words: remotal sets, Hilbert space, Hilbert space operators, farthest point

Received: May 14, 2012

© 2012 Academic Publications

§Correspondence author

1. Introduction

Let X be a Banach space and E be a closed bounded set of X. For $x \in X$, let

$$D(x, E) = \sup_{e \in E} \| x - e \| .$$

This supremum needs not to be attained. If $\forall x \in X$, there exists some $e \in E$ such that $D(x, E) = \| x - e \|$, then E is called remotal. For $x \in X$, we set

$$F(x, E) = \{e \in E : \| x - e \| = D(x, E)\}.$$

The point e is called a farthest point of E from x. The study of remotal sets goes back to the sixties. Edelstein [1] showed that if X is a uniformly convex Banach space, then the set of points in X that has farthest points in E is dense in X.

The importance of the study of remotal sets comes from its ties to geometry of Banach spaces. Such study is little difficult and less developed. On the contrary, closest points are well studied and well established.

The problem of remotality in $L^p(I, X), 1 \leq p < \infty$, was initiated by Khalil and Al-Sharif [4]. Then it was developed by Sababheh, and Khalil [2]. One of the standing conjectures in the theory of remotal sets is "Every uniquely remotal set in a Banach space is a singleton". So much work was done to solve such a conjecture (see [5] and [3]).

In this paper, we propose some new results on remotality in the space of bounded linear operators on Hilbert spaces.

Throughout this paper, H denotes a separable Hilbert space. If E is any subset of H, then $E^{\perp} = \{x \in H : < e, x >= 0 \, \forall e \in E\}$, the annihilator of E.

2. Remotal Sets in Certain Classical Spaces

In this section we introduce new classes of remotality of some spaces.

For X a normed space, and $E \subseteq X$, we let $\ell^1(E) = \{(x_n); x_n \in E$, and $\sum \| x_n \| < \infty\}$. Similarly for $c_0(E), \ell^p(E)$ and $\ell^\infty(E)$. For a Banach space X, let us denote by $B_1[X]$ the closed unit ball of X.

Theorem 1. *Let E be a finite set in a Banach space X. Then $\ell^1(E)$ is remotal in $\ell^1(X)$ if and only if $E = \{0\}$.*

Proof. If $E = \{0\}$, then $\ell^1(E) = \{0\}$, is the zero sequence. Hence $\ell^1(E)$ is a finite set.

So $\ell^1(E)$ is remotal in $\ell^1(X)$.

Conversely, suppose E is finite $E \neq \{0\}$ and that $\ell^1(E)$ is remotal in $\ell^1(X)$.

Since E is finite, hence being compact, E is remotal in X.

Let $e \in F(0, E)$ and $f = (0, 0, 0, \ldots, 0, \ldots) \in \ell^1(X)$.

If $e \neq 0$, then $(e, e, e, \ldots, e, 0, 0, 0, \ldots) \in \ell^1(E)$, where e appears in the first n-coordinates.

But in such a case we have:

$$\| (e, e, e, \ldots, e, 0, 0, \ldots) - f \| = n \| e \| .$$

This implies that $D(f, \ell^1(E)) = \infty$ and $\ell^1(E)$ is not remotal.
Consequently, e must equal to zero and $E = \{0\}$. □

In a similar way one can prove:

Proposition 1. *Let E be a finite set in a Banach space X. Then $\ell^p(E)$ is remotal in $\ell^p(X)$ if and only if $E = \{0\}$.*

Proposition 2. *Let E be a finite set in a Banach space X. Then $c_0(E)$ is remotal in $c_0(X)$ if and only if $E = \{0\}$.*

Let us recall some definitions from the literature.

1. Let X be a Banach space, $Y \subseteq X$. Y is called 1- summand if there is $W \subseteq X$ such that $X = Y \oplus_1 W$ where if $x \in X$ then
   $x = y + \omega$, with $y \in Y, w \in W$ and $\| x \| = \| y \| + \| w \|$.

2. Let X be a Banach space, $Y \subseteq X$. Y is called p- summand if there is $W \subseteq X$ such that $X = Y \oplus_p W$ where if $x \in X$ then $x = y + \omega$, where $y \in Y, w \in W$ and
   $\| x \|_p = (\| y \|^p + \| w \|^p)^{\frac{1}{p}}$.

3. Let X be a Banach space, $Y \subseteq X$. Y is called ∞- summand if there is $W \subseteq X$ such that $X = Y \oplus_\infty W$ where if $x \in X$ then $x = y + \omega$, $y \in Y$, and $w \in W$ and
   $\| x \|_\infty = \max(\| y \|, \| w \|)$.

Proposition 3. *Let Y be a p-summand of a Banach space X, where $1 \leq p < \infty$. Then $B_1[Y]$ is remotal in X.*

Proof. Suppose that $X = Y \oplus_p W$ and $x = y + w$. This implies that $\| x \|_p = (\| y \|^p + \| w \|^p)^{\frac{1}{p}}$.

Since $B_1[Y]$ is remotal in Y, let $\hat{y}$ be a farthest element in $B_1[Y]$ from y. Then,

$$\begin{aligned} \| x - \hat{y} \|_p^p &= \| y - \hat{y} + w \|^p \\ &= \| y - \hat{y} \|^p + \| w \|^p \\ &\geq \| y - z \|^p + \| w \|^p \quad \forall z \in B_1[Y] \\ &= \| y + w - z \|_p^p \quad \forall z \in B_1[Y] \\ &= \| x - z \|_p^p \quad \forall z \in B_1[Y]. \end{aligned}$$

Consequently, $\hat{y}$ is a farthest element in $B_1[Y]$ from $x = y + w$. □

Similarly one can prove:

Proposition 4. *Let Y be an ∞- summand of a Banach space X. Then $B_1[Y]$ is remotal in X.*

Definition 2. Let X be a Banach space. Then for $E \subseteq X$, we let

$$L(B_1[X], E) = \{T \in L(X, X) : T(B_1[X]) \subseteq E\}.$$

Let us denote $L(B_1[X], E)$ by J.

Proposition 5. *If E is a finite set in a Banach space X. Then $L(B_1[X], E)$ is remotal in $L(X, X)$ if $0 \in E$.*

Proof. Let $E = \{x_1, x_2, \ldots, x_n\}$, and $T \in L(X, X)$.

Assume that $D(T, J) = r$. Hence there exists a sequence $T_n \in J$ such that $\lim_{n\to\infty} \| T_n - T \| = r$.

Now, $B_1[X]$ is convex. Hence $T_n(B_1[X])$ is convex. Thus, since E is finite $T_n(B_1[X])$ is a singleton. But also, $0 \in B_1[X]$ and $T_n 0 = 0 \, \forall n$.

Hence if $0 \notin E$ then $L(B_1[X], E) = \phi$, and so remotal (being compact).

If $0 \in E$, then $T_n(B_1[X]) = \{0\}$ $\forall N$, and so $T_n = 0 \forall n$, and $L(B_1[X], E)$ is remotal in $L(X, X)$. □

Proposition 6. *If E is a finite set in a Banach space X. Then $C(I, E)$ is remotal in $C(I, X)$.*

Proof. Since E is finite and I is connected, then $g \in C(I,E)$ if and only if f is constant. Hence for $f \in C(I,X)$ we have $D(f, C(I,E)) = \sup_{g\in C(I,E)} \| f - g \|$. But $C(I,E)$ is a finite set. Hence $C(I,E)$ is remotal in $C(I,X)$. □

Proposition 7. *Let E be a compact uniquely remotal set in a Banach space X and $H : X \to E$ be defined by $H(x) = F(x,E)$. If H is continuous, then $C(I,E)$ is remotal in $C(I,X)$.*

Theorem 3. *If Y is a closed subspace of a finite dimensional Hilbert space, then the set $B_1[L(H,Y)]$ is remotal in $L(H,H)$.*

Proof. Let $T \in L(H,H)$ and $D(T, B_1[L(H,Y)]) = r$. Hence there exists T_n in $B_1[L(H,Y)]$ such that $\lim\limits_{n\to\infty} \| T_n - T \| = r$.

Since Y is finite dimensional, then $L(H,Y)$ is a reflexive space. Thus by the Alaoglu theorem $B_1[L(H,Y)]$ is w^*-compact. So there exists a subsequence T_{nk} such that $T_{nk} \to^{w*} A \in B_1[L(H,Y)]$ and this implies that $T_{nk} - T \to^{w*} T - A$.

This means that

$$\lim_{n\to\infty} |\langle (T_{nk} - T)x, y\rangle| = |\langle (T - A)x, y\rangle|$$

$(\forall y \in (B_1[L(H,Y)])^* = B_1[L(H,Y)])$

$$\leq \ \| \ (T - A)x \| \| \ y \| = \| \ T - A \|,$$

$$\sup_{\|y\|=1} |\langle (T_{nk} - T)x, y\rangle| \leq \| \ T - A \|,$$

$$\| \ (T_{nk} - T)x \| \leq \| \ T - A \| \ .$$

Now,

$$\sup_{\|x\|=1} \| \ T_{nk} - T)x \| = \| \ T_{nk} - T \| \leq \| \ T - A \|$$

$$r = \lim_{nk\to\infty} \| \ T - T_{nk} \| \leq \| \ T - A \| \ .$$

Hence,

$$D(T, B_1[L(H,Y)]) = \| \ T - A \| \ .$$

□

3. Remotality of $L(B_1(H), B_1(Y))$ in $L(H,H)$

Let H be a separable Hilbert space and Y be a closed subspace of H. So $H = Y \oplus Y^{\perp}$. Set $L(H,H)$ to denote the space of bounded linear operators on H, and $E = L(B_1(H), B_1(Y))$, the space of all bounded linear operators that takes the unit ball of H into the unit ball of Y. Then E is a closed convex bounded subset of $L(H,H)$.

In this section, we want to discuss remotality of E in $L(H,H)$.

Remark: Any operator $T : H \to H$ has a matrix representation

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} : Y \oplus Y^{\perp} \to Y \oplus Y^{\perp},$$

where

$$T(y+\hat{y}) = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} y \\ \hat{y} \end{bmatrix},$$

so

$$T_1 : Y \to Y, \qquad T_2 : Y^{\perp} \to Y,$$

$$T_3 : Y \to Y^{\perp}, \qquad T_4 : Y^{\perp} \to Y^{\perp}.$$

Definition 4. An operator $T \in L(H,H)$ is called diagonal, if $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix}$.

A well known lemma about the diagonal operators is the following:

Lemma 1. *If $T \in L(H,H)$ and $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, then*

$$\| T \| = \max\{\| A \|, \| B \|\}.$$

Proof. Let $T \in L(H,H)$ such that $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Then,

$$\begin{aligned}
\| T \| &= \sup_{\|x\|=1} \| Tx \| \\
&= \sup_{\|y+\hat{y}\|=1} \| T(y+\hat{y}) \| \\
&= \sup_{\|y+\hat{y}\|=1} \left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} y \\ \hat{y} \end{bmatrix} \right\| \\
&= \sup_{\|y+\hat{y}\|=1} \left\| \begin{bmatrix} Ay \\ B\hat{y} \end{bmatrix} \right\|
\end{aligned}$$

$$\leq \max \begin{bmatrix} \sup\limits_{\|y+\hat{y}\|=1} & \| Ay \| \\ \sup\limits_{\|y+\hat{y}\|=1} & \| B\hat{y} \| \end{bmatrix},$$

and for $\hat{y} = 0$ we have

$$\leq \max \begin{bmatrix} \| A \| \\ \| B \| \end{bmatrix}.$$

Now,

$$\| A \| \leq \| T \|$$

and

$$\| B \| \leq \| T \| .$$

Thus,

$$\max\{\| A \|, \| B \|\} \leq \| T \| .$$

This ends the proof. $\square$

Theorem 5. *The set E is remotal with respect to diagonal operators in $L(H, H)$.*

Proof. Let T be a diagonal matrix. Then $T = T_1 + T_2$, where $T_1 : Y \to Y$ and $T_2 : Y^{\perp} \to Y^{\perp}$.

Now, T_1 has the farthest point in $L(Y, B_1(Y))$, say $A = -\dfrac{T_1}{\| T_1 \|}$.

Let $\hat{T} = A + T_2$. Then $\| T - \hat{T} \| \geq \| T - S \| \; \forall S \in E$.

Thus, E is remotal in the diagonal operators in $L(H, H)$. $\square$

Definition 6. An operator $T \in L(H,H)$ is called of Schmidt type if for every orthonormal basis (e_n) of H, there is an orthogonal sequence (a_n) in H such that

$$Tx = \sum_{n=1}^{\infty} \langle a_n, x\rangle e_n.$$

Example. Every unitary operator is of Schmidt type.

Theorem 7. *Let $T \in L(H,H)$ be of Schmidt type, and Y be any closed subspace of H. Then there exists $A \in B_1(L(H,Y))$ such that*

$$\| T - A \| = D(T, B_1, (L(H,Y))).$$

Proof. Let $H = Y \oplus Y^{\perp}$, and (θ_n) be a basis for Y, and (f_n) be a basis for $Y^{\perp}$. Since T is of Shcmidt type, there is (a_n) orthogonal sequence in H such that

$$T = \sum_{n=1}^{\infty} a_n \otimes z_n,$$

where

$$z_{2n} = \theta_n, \quad z_{2n-1} = f_n \quad \forall n \geq 1,$$

so

$$T = \sum_{n=1}^{\infty} a_{2n} \otimes \theta_n + \sum_{n=1}^{\infty} a_{2n-1} \otimes f_n,$$

where

$$\begin{aligned} Tx &= \sum_{n=1}^{\infty} \langle a_{2n}, x\rangle \theta_n + \sum_{n=1}^{\infty} \langle a_{2n-1}, x\rangle f_n \\ &= T_1 x \qquad + \qquad T_2 x \quad , \end{aligned}$$

and

$$\| T \| = \max\{\| T_1 \|, \| T_2 \|\}$$

Now, $T_1 \in L(H,Y)$. Hence the operator $B = \dfrac{-T_1}{\| T_1 \|} \in B_1(L(H,Y))$ and

$$\| T_1 - B \| \geq \| T_1 - S \| \qquad \forall S \in B_1(L(H,Y)).$$

Now,

$$T - B = (T_1 - B) + T_2$$

and

$$\begin{aligned}\| T - B \| &= \max\{\| T_1 - B \|, \| T_2 \|\} \\ &\geq \max\{\| T_1 - S \|, \| T_2 \|\} \\ &= \| T - S \| \quad \forall S \in B_1(L(H,Y)).\end{aligned}$$

Hence $B_1(L(H,Y))$ is remotal with respect to Schmidt operators. $\square$

References

[1] M. Edelstein, Farthest points of sets in uniformly convex Banach spaces, *Israel Journal of Mathematics*, **4**, No. 3 (1966), 171-176.

[2] M. Sababheh, R. Khalil, Remotality of closed bounded convex sets in reflexive spaces, *Numerical Functional Analysis and Optimization*, **29** (2008), 1166-1170.

[3] T.D. Narang, On singletonness of uniquely remotal sets, *Periodica Mathematica Hungarica*, **21**, No. 1 (1990), 17-19.

[4] R. Khalil, Sh. Al-Sharif, Remotal sets in vector valued function spaces, *Scientiae Mathematicae Japonica*, **63** (2006), 433-441.

[5] A.A. Astaneh, On uniquely remotal subsets of Hilbert spaces, *Indian J. Pure Appl. Math.*, **14** (1983), 1217-1219.