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DUALITY BETWEEN THE GENERALIZED CANONICAL POLYNOMIALS AND SOME SPECIAL DETERMINANTS

Mohamed K. El-Daou

Applied Sciences Department
College of Technological Studies
P.O. Box 64287, Shuwaikh/B, 70453, KUWAIT
e-mail: mk.eldaou@paaet.edu.kw

Abstract: Given a linear differential operator D and a set of polynomials $\{V_k(x); k \in \mathbf{N};\ \deg[V_k] = k\}$, Ortiz [8] has proved that there exists an infinite set W in $\mathbf{N}$ and a sequence of polynomials, $\{Q_k(V); k \in W\}$, called *generalized canonical polynomials*, such that for all $k \in W$, $DQ_k = V_k + r_k$ where $r_k \in \operatorname{span}\{V_i; i \notin W\}$. In this paper we make use of the concept of biorthogonality to identify the linear functionals forming with $\{Q_k(V); k \in W\}$ a biorthogonal system. As a result of this identification, each $Q_k(V)$ will be given in a determinant form. We also show that these Q_k's can be used to evaluate some determinants with polynomial entries and defined by ordinary differential equations.

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1. Introduction

The concept of biorthogonality is a generalization of the notion of orthogonality in Hilbert spaces which itself comes from the notion of orthogonality for functions and polynomials. The interest in the concept of biorthogonality has been observed in several references (see e.g [1], [2], [3] and the references given therein). In Davis [3] the biorthogonality was used to classify some in-

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terpolation methods in terms of linear functionals. Brezinski, in [1]-[2], gave a comprehensive discussion of the biorthogonality with interesting applications which include differential equations.

In [5], El-Daou and Ortiz presented a similar analysis for a class of methods that approximate differential equations: We used the notion of biorthonormality in the discussion of structural relations between different numerical methods based on the truncated polynomial expansions. Among those methods is the Tau method that was invented by Lanczos in [7] to solve simple first order differential equations. With the Lanczos Tau method, the approximate solution is given in terms of a special polynomial basis called canonical polynomials $\{Q_k(x); k \in W\}$ where W is an infinite set of $\mathbf{N}$. The Tau method was developed later by Ortiz in [8]-[9] to treat problems of different complexities. With Ortiz's approach of the Tau method, we consider an arbitrary set of polynomials $\{V_k(x); k \in \mathbf{N}\}$ and express the desired approximate solution in terms of a *generalized* canonical polynomials basis denoted by $\{Q_n(V); n \in W\}$. A self-starting recursive formula to generate those Q_n's is given in [8].

In this paper, we make use of the general interpolation Newton's formula, as stated in [3], to show that the generalized canonical polynomials $\{Q_n(V); n \in W\}$ are *biorthonormal*, in the sense given below, to a set of linear functionals related to a given differential operator. As a result of this, each generalized canonical polynomial $Q_n(V)$ will be given a determinant representation. Further we show how the canonical polynomials can be used to evaluate special determinants with polynomial entries. This is illustrated through the evaluation of special forms of Hankel's determinants.

2. Biorthonormality and Canonical Polynomials

Let $\mathcal{D}$ be the class of linear differential operators with polynomial coefficients acting on $X := \mathcal{C}^{\nu}[a,b]$, the space of ν–times continuously differentiable functions. Let $D \in \mathcal{D}$ be given as

$$(Dy)(x) := a_{\nu}(x)y^{(\nu)}(x) + a_{\nu-1}(x)y^{(\nu-1)}(x) + \cdots + a_0(x)y(x), \quad x \in [a,b],$$

where $\nu \geq 1$, the coefficients $\{a_k(x)\}_{k=0}^{\nu}$ are polynomials, and $y^{(k)}(x)$ stands for the kth derivative of y with respect to x.

Let us consider a sequence of polynomials $\{V_k(x); k \in \mathbf{N}\}$ such that $\deg[V_k(x)] = k$ for all $k \in \mathbf{N}$ and

$$V_k(x) = \sum_{i=0}^{k} v_{ki}x^i \ ; \quad v_{kk} \neq 0.$$

We call *kth generating polynomial*, the polynomial $P_k(V)$ given by

$$P_k(V) := DV_k(x) = \sum_{i=0}^{d_k} P_{ki} V_i \; ; \;\; d_k \in \mathbf{N}, \;\; \{P_{ki}\} \subset \mathbf{R}. \tag{1}$$

Let

$$\mathbf{M}(D, \underline{V}) := (P_{ij})_{i,j \in \mathbf{N}} = \begin{pmatrix} P_{00} & P_{01} & \cdots \\ P_{10} & P_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Since D has polynomial coefficients, $\mathbf{M}(D, \underline{V})$ is bounded-from-above; that is, there exists an integer $k \in \mathbf{Z} := \{0, \pm 1, \pm 2, ...\}$ such that

$$P_{ij} = 0 \quad \text{whenever } (j-i) \geq k+1; \;\; i, j \geq 0. \tag{2}$$

We associate to D an integer $h(D)$ given by

$$h(D) := \inf\{k;\; k \text{ satisfies } (2)\},$$

which is called *height* of D (see [8]). Let

$$W(D) := \{i \in \mathbf{N};\; P_{i\,h(D)+i} \neq 0\}. \tag{3}$$

$W(D)$ is called *disconnected* in $\mathbf{N}$ if there exist p subsets $\{W_i\}_{i=1}^{p} \subset \mathbf{N}$, $(p \geq 2)$, such that

$$W(D) = \bigcup_{i=1}^{p} W_i \quad \text{and} \quad \sup(W_i) < m_i < \inf(W_{i+1}), \tag{4}$$

for some p distinct $\{m_i \in \mathbf{N}, i = 1, 2, \ldots, p\}$; otherwise $W(D)$ is called *connected.*

Let π_k stand for the linear projection that associates to each element in $\text{span}\{V_i; i \in \mathbf{N}\}$ its kth coefficient; that is

$$\pi_k : \sum_{i \in \mathbf{N}} \lambda_i V_i \longmapsto \lambda_k \in \mathbf{R}, \;\; (\forall k \in \mathbf{N}). \tag{5}$$

From Ortiz [8], every $D \in \mathcal{D}$ is uniquely associated with a sequence $\underline{Q} := \{Q_k(V);\; k \in W(D)\}$ of *canonical polynomials* defined by

$$DQ_k = V_k + r_k(V),$$

where $r_k := r_k(V) \in R_W := \text{span}\{V_i; i \notin W(D)\}$. A self starting, recursive formula for constructing the elements of $\underline{Q}$ is given in [8].

In this section we make use of the general interpolation Newton's formula, (see Davis [3]), to show that the canonical polynomials $\{Q_k(V); k \in W(D)\}$ are *biorthonormal* to linear functionals related to D and to the projections $\{\pi_k;\ k \in \mathbf{N}\}$. We also show that each $Q_k(V)$ has a determinant form. In order to formulate our results we need to remind some basic notation.

Setting $L_k := \pi_k \circ D$, where π_k is defined by (5), when $n > k$, we introduce the arrays

$$\underline{V_k^n} := (V_k, V_{k+1}, \ldots, V_n), \quad \underline{L_k^n} := (L_k, L_{k+1}, \ldots, L_n),$$
$$G[\underline{L_k^n}, \underline{V_k^n}] := \det\left[L_i(V_j)\right]_{i,j=k}^n .$$

Definition 1. $\underline{V_0^n}$ and $\underline{L_0^n}$ define a biorthonormal system if $L_i(V_j) = \delta_{ij}$ for all $i, j = k, k+1, \ldots, n$, where δ_{ij} denotes the Kronecker symbol.

Lemma 2. *If $\underline{V_k^n}$ and $\underline{L_k^n}$ are independent, then*

$$G[\underline{L_k^n}, \underline{V_k^n}] \neq 0. \tag{6}$$

Conversely, if either $\underline{V_k^n}$ or $\underline{L_k^n}$ is independent and (6) holds then the other set is also independent.

As mentioned above, for any $D \in \mathcal{D}$, the set $W(D)$, defined by (3), has a finite complement in $\mathbf{N}$ (Ortiz [8]). This means that $W(D)$ is not necessarily connected in $\mathbf{N}$, but has always an infinite connected component in it. We will consider first the case of connected $W(D)$ and thereafter deduce the results for the disconnected case. For simplicity, $W(D)$ (resp. $h(D)$) will be indicated by W (resp. h).

Lemma 3. *Let us assume that $D \in \mathcal{D}$ and that W is connected; let $\kappa := \inf[W]$ and let $X_\kappa := span\{V_n; n \geq \kappa\}$. Then we have*

$$L_{n+h}\left(Q_{m+h}\right) = \delta_{mn}, \quad \forall m, n \geq \kappa. \tag{7}$$

Proof. Let m and $n \geq \kappa$. From the definitions of $\{L_k\}$ and $\{Q_k\}$ we have

$$L_{n+h}\left(Q_{m+h}\right) = \pi_{n+h}[DQ_{m+h}] = \pi_{n+h}\left[V_{m+h} + r_{m+h}\right].$$

When $m = n$, we have $\pi_{n+h}\left[V_{m+h} + r_{m+h}\right] = 1$ because $r_{m+h} \in R_W := \operatorname{span}\{V_i; i \notin W\}$; i.e. r_{m+h} has no nonzero coefficient along V_{n+h} for any

$n \in W$. On the other hand, if $n < m$ and $L_{n+h}(Q_{m+h}) \neq 0$, we can find a nonzero $\alpha \in \mathbf{R}$ such that $\pi_{n+h}[V_{m+h} + r_{m+h}] = \alpha$. This implies that r_{m+h} can be written as

$$r_{m+h}(V) = \alpha V_{n+h} + \hat{r}_{m+h}(V); \quad \hat{r}_{m+h}(V) \in R_W,$$

which contradicts the facts that $r_{m+h} \in R_W$ and $n \in W$. $\square$

We can identify now, by means of Lemmas 2 and 3, the set of linear functionals that are biorthonormal to the generalized canonical polynomials:

Theorem 4. *Suppose that the assumptions of Lemma 3 hold true. Then there exist uniquely determined sequences*

$$\{Q^*_{h+m}; m \geq \kappa\} \subset X_\kappa \quad \text{and} \quad \{L^*_{h+m}; m \geq \kappa\}$$

given by

$$Q^*_{h+m} = \frac{1}{G[\underline{L^{h+m-1}_{h+\kappa}}, \underline{V^{m-1}_{\kappa}}]} \begin{vmatrix} P_{\kappa\ \kappa+h} & P_{\kappa+1\ \kappa+h} & \dots & P_{m\ \kappa+h} \\ 0 & P_{\kappa+1\ \kappa+h+1} & \dots & P_{m\ \kappa+h+1} \\ 0 & 0 & & \dots \\ \vdots & \vdots & \ddots & \dots \\ 0 & \dots & \dots & P_{m\ m+h-1} \\ V_\kappa & V_{\kappa+1} & \dots & V_m \end{vmatrix} \tag{8}$$

and $L^*_{h+m} = \frac{L_{h+m}}{P_{m\ h+m}}$, *such that* $Q_{h+m} = \frac{1}{P_{m\ m+h}} Q^*_{h+m}$ *for all* $m \geq \kappa$.
Further, for all $y \in$*span*$\{V_i; i \geq \kappa\}$ *we have*

$$y = \sum_{i=\kappa}^{\kappa+m} L^*_{h+i}(y) Q^*_{h+i} = \sum_{i=\kappa}^{\kappa+m} L_{h+i}(y) Q_{h+i}.$$

Proof. Let $m \in \mathbf{N}$, SV:=$\{V_{\kappa+i}; 0 \leq i \leq m-\kappa\}$ and SL:=$\{L_{h+\kappa+i}; 0 \leq i \leq m-\kappa\}$. Let us show first that SV and SL are independent. Given i and $j \geq 0$, we write

$$\begin{aligned} L_{h+\kappa+i}[V_{\kappa+j}] &= \pi_{h+\kappa+i}[DV_{\kappa+j}] == \pi_{h+\kappa+i}[\sum_{l=0}^{\kappa+j+h} P_{\kappa+j,l} V_l] \\ &= \begin{cases} P_{\kappa+j\ \kappa+j+h} & if i = j \\ 0 & if i¿j \end{cases}. \end{aligned}$$

This implies that

$$G[\underline{L_{h+\kappa}^{h+m-1}}, \underline{V_{\kappa}^{m-1}}] = P_{\kappa\ \kappa+h} \times P_{\kappa+1\ \kappa+h+1} \times P_{m-1\ m+h-1} \neq 0,$$

and, therefore, by Lemma 2, SV and SL are independent. Hence SV and SL satisfy the conditions of Generalized Newton's Interpolation formula (see [3]). From the latter two sequences can be found, $\{Q_{h+i}^*; i \geq \kappa\} \subset X_\kappa$ given by (8) and $\{L_{h+i}^*; i \geq \kappa\} \subset \mathcal{L}(X_\kappa)$ given as

$$\begin{aligned} L_{h+m}^* &= \frac{1}{G[\underline{L_{h+\kappa}^{h+m}}, \underline{V_{\kappa}^{m}}]} \begin{vmatrix} P_{\kappa\ h+\kappa} & 0 & \dots & 0 \\ P_{\kappa+1\ h+\kappa} & P_{\kappa+1\ h+\kappa+1} & 0 & \vdots \\ \vdots & \vdots & \ddots & \\ P_{m-1\ h+m-1} & & P_{m-1\ h+m-1} & 0 \\ L_{h+\kappa} & L_{h+\kappa+1} & \dots & L_{h+m} \end{vmatrix} \\ &= \frac{L_{h+m}}{P_{m\ h+m}}, \end{aligned}$$

such that

$$L_{m+h}^*(Q_{n+h}^*) = L_{m+h}\left(\frac{Q_{n+h}^*}{P_{m\ m+h}}\right) = \delta_{mn} \ \ \forall m, n \geq \kappa.$$

On account of (7) it follows that $Q_{m+h} = \dfrac{1}{P_{m\ m+h}} Q_{m+h}^*, \ \forall m \geq \kappa.$ □

When W is disconnected we obtain a similar result:

Theorem 5. *Let $D \in \mathcal{D}$ and assume that W is disconnected in the sense of (4), $\kappa := \inf(W)$. Then there exists a uniquely determined sequence $\{Q_{h+m}^*; m \in W\} \subset span\{V_m; m \in W\}$ given as*

$$Q_{h+m}^* = \frac{1}{G_W[\underline{L_{h+\kappa}^{h+m-1}}, \underline{V_{\kappa}^{m-1}}]} det \begin{bmatrix} \mathbf{M}_W^{h+m} \\ \underline{V_W^{\kappa+m}} \end{bmatrix}, \tag{9}$$

where

$$G_W[\underline{L_{h+k}^{h+m}}, \underline{V_k^m}] := det\,[L_{h+i}(V_j)]_{\substack{i,j=k \\ i,j \in W}}^m, \quad \mathbf{M}_W^{h+m} := [P_{i\ h+j}]^T_{\substack{i=\kappa,\kappa+1,\dots,m \\ j=\kappa,\kappa+1,\dots,m-1 \\ i,j \in W}},$$

and $\underline{V_W^{\kappa+m}} := (V_i)_{\substack{i=\kappa,\kappa+1,\dots,m \\ i \in W}}$, such that the canonical polynomials associated with D are given by

$$Q_{h+m} = \frac{1}{P_{m\ h+m}} Q_{h+m}^* \ \ \forall m \in W.$$

Let us consider two examples that illustrate the generation of the canonical polynomials using the results of this section.

Example 1. Let us consider the differential operator D

$$Dy(x) := (2+2x^3)y^{''}(x) - \frac{43}{5}(1+x^2)y^{'}(x) + 3xy(x).$$

Take $V_k = x^k$, for all $k \in \mathbf{N}$. We wish to construct the canonical polynomials associated with D using Theorem 5. From (1),

$$P_k(x) := Dx^k = \left(-2k+2k^2\right)x^{k-2} - \frac{43k}{5}x^{k-1} + \left(3 - \frac{53k}{5} + 2k^2\right)x^{k+1}.$$

Clearly $h=1$, $W=\{0,1,2,3,4\}\cup\{k;k\geq 6\}$; $\kappa := \inf(W) = 0$ and

$$P_{kj} = \begin{cases} -2k+2k^2 & \text{if j=k-2} \\ -43k/5 & \text{if j=k-1} \\ 5-53k/5+2k^2 & \text{if j=k+1} \\ 0 & \text{otherwise.} \end{cases}$$

Now by means of (9) we find $Q^*_{m+1}(x)$ for all $m \in W$:

For $m=1$,

$$Q^*_2(x) = \frac{1}{G_W[\underline{L^1_1}, \underline{x^0_0}]}\begin{vmatrix} 3 & 0 \\ 1 & x \end{vmatrix} \qquad \text{where} \quad G_W[\underline{L^1_1}, \underline{x^0_0}] := 3,$$

which gives $Q^*_2(x) = 3x$ and $Q_2(x) = \frac{1}{P_{1,2}}Q^*_2(x) = \frac{1}{-28/5}Q^*_2(x) = \frac{5}{-28}x$.

For $m=6$,

$$Q^*_7(x) = \frac{1}{G_W[\underline{L^6_1}, \underline{x^5_0}]}\begin{vmatrix} 3 & 0 & -86/5 & 12 & 0 & 0 \\ 0 & -28/5 & 0 & -129/5 & 24 & 0 \\ 0 & 0 & -51/5 & 0 & -172/5 & 0 \\ 0 & 0 & 0 & -54/5 & 0 & 60 \\ 0 & 0 & 0 & 0 & -37/5 & -258/5 \\ 1 & x & x^2 & x^3 & x^4 & x^6 \end{vmatrix};$$

$$G_W[\underline{L^6_1}, \underline{x^5_0}] := \begin{vmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & -28/5 & 0 & 0 & 0 \\ -86/5 & 0 & -51/5 & 0 & 0 \\ 12 & -129/5 & 0 & -54/5 & 0 \\ 0 & 24 & -172/5 & 0 & -37/5 \end{vmatrix}.$$

Proceeding this way we obtain,

$$Q_1(x) = \frac{1}{3}, \;\; Q_2(x) = -\frac{5x}{28}, \;\; Q_3(x) = \frac{-86}{153} - \frac{5x^2}{51},$$
$$Q_4(x) = \frac{10}{27} + \frac{215x}{504} - \frac{5x^3}{54},$$
$$Q_5(x) = \frac{14792}{5661} - \frac{150x}{259} + \frac{860x^2}{1887} - \frac{5x^4}{37}$$
$$Q_7(x) = \frac{3187336}{322677} - \frac{431075x}{88578} + \frac{73960x^2}{35853} + \frac{250x^3}{513} - \frac{430x^4}{703} + \frac{5x^6}{57},$$
$$Q_8(x) = \frac{-1035440}{126429} + \frac{4500x}{2479} - \frac{60200x^2}{42143} + \frac{1050x^4}{2479} + \frac{5x^7}{134}.$$

After some algebraic manipulations it follows that the determinant satisfies the same recursive relation as canonical polynomials; namely

$$Q_{k+1}(x) = \frac{5}{15-53k+10k^2}\left[x^k - (2k^2-2k)Q_{k-2}(x) + \frac{43k}{5}Q_{k-1}(x)\right] \; (k \geq 0).$$

Example 2. Let D be a second order differential operator with constant coefficients A and B:

$$Dy(x) = y^{''}(x) + Ay^{'}(x) + By(x).$$

Let us consider the case of $V_k = T_k(x)$, Chebyshev polynomials of degree $k \geq 0$ with $x \in [-1, 1]$. These are defined as $T_k(x) = \cos(k \arccos x)$:

$$T_0(x) = 1, \; T_1(x) = x, \; \text{and} \; T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \quad k \geq 1.$$

The first and second derivatives are

$$T^{'}_{2n} = 4n\sum_{k=0}^{n-1} T_{2k+1}, \qquad T^{'}_{2n+1} = (2n+1)T_0 + 2(2n+1)\sum_{k=1}^{n} T_{2k},$$
$$T^{''}_{2n} = 4n^3T_0 + 8n\sum_{k=1}^{n-1}(n^2-k^2)T_{2k}, \;\; T^{''}_{2n+1} = 4(2n+1)\sum_{k=0}^{n-1}(n-k)(n+1+k)T_{2k+1}.$$

Let us generate the generalized canonical polynomials of D that correspond to the Chebyshev basis by applying Theorem 5. From (1),

$$P_{2n}(x) \;\; = \;\; 4n^3T_0 + 8n\sum_{k=1}^{n-1}(n^2-k^2)T_{2k} + 4An\sum_{k=0}^{n-1} T_{2k+1} + BT_{2n},$$

$$\begin{aligned}P_{2n+1}(x) &= 4(2n+1)\sum_{k=0}^{n-1}(n-k)(n+1+k)T_{2k+1} + A(2n+1)T_0 + \\ & 2(2n+1)\sum_{k=1}^{n} T_{2k} + BT_{2n+1}.\end{aligned}$$

Clearly $h = 0$, $W = \mathbf{N}$ and the entries of (8) are

$$\begin{array}{ll}
P_{2n,0} = 4n^3 & \text{if } j = 0 \\
P_{2n,2j} = 8n(n^2 - j^2) & \text{if } j = 1, 2, ..., n-1 \\
P_{2n,2j+1} = 4nA & \text{if } j = 0, 1, ..., n-1 \\
P_{2n,2n} = B & \text{if } j = 2n
\end{array}$$

$$\begin{array}{ll}
P_{2n+1,0} = (2n+1)A & \text{if } j = 0 \\
P_{2n+1,2j+1} = 4(2n+1)(n-j)(n+1+j) & \text{if } j = 0, 1, 2, ..., n-1 \\
P_{2n+1,2j+1} = 2A(2n+1) & \text{if } j = 1, 2, ..., n \\
P_{2n+1,2n+1} = B & \text{if } j = 2n+1.
\end{array}$$

Therefore, we obtain, for instance

$$\begin{aligned}
Q_1^*(x) &= \frac{1}{B}\begin{vmatrix} B & A \\ T_0 & T_1 \end{vmatrix} = \frac{1}{B}(-AT_0 + BT_1), \\
Q_2^*(x) &= \frac{1}{B^2}\begin{vmatrix} B & A & 4 \\ 0 & B & 4A \\ T_0 & T_1 & T_2 \end{vmatrix} = \frac{1}{B^2}\left((4A^2 - 4B)T_0 - 4ABT_1 + B^2T_2\right), \\
Q_7^*(x) &= \frac{1}{B^7}\begin{vmatrix}
B & A & 4 & 3A & 32 & 5A & 108 & 7A \\
0 & B & 4A & 24 & 8A & 120 & 12A & 336 \\
0 & 0 & B & 6A & 48 & 10A & 192 & 14A \\
0 & 0 & 0 & B & 8A & 80 & 12A & 280 \\
0 & 0 & 0 & 0 & B & 10A & 120 & 14A \\
0 & 0 & 0 & 0 & 0 & B & 12A & 168 \\
0 & 0 & 0 & 0 & 0 & 0 & B & 14A \\
T_0 & T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7
\end{vmatrix}.
\end{aligned}$$

Again, some algebraic manipulations show that the above determinants satisfy the same recursion given in [8],

$$\begin{aligned}
Q_{2n} &= B^{-1}[T_{2n} - 8n\sum_{k=1}^{n-1}(n^2 - k^2)Q_{2k} - 4An\sum_{k=0}^{n-1}Q_{2k+1} - 4n^3Q_0], \\
Q_{2n+1} &= B^{-1}[T_{2n+1} - 4(2n+1)\sum_{k=0}^{n-1}(n-k)(n+1+k)Q_{2k+1}] -
\end{aligned}$$

$$B^{-1}[2(2n+1)\sum_{k=1}^{n} Q_{2k} - A(2n+1)Q_0].$$

The above discussion suggests that if a determinant is defined as a solution of a differential equation then it becomes possible to evaluate it in a recursive manner. This is explained in the following section.

3. Evaluation of Special Forms of Hankel's Determinants

Here we make use of the generalized canonical polynomials to evaluate Hankel's determinants,

$$H_0(n,x) = \det\left[a_{i+j}(x)\right]_{0\le i,j\le n},$$

where n is a fixed in $\mathbf{N}$ and $\{a_k(x)\}$ are polynomial given as

$$a_k(x) = \sum_{m=0}^{k} \binom{3k-m}{m} x^m.$$

Some results concerning the evaluation of $H_0(n,x)$ can be found in [6] and more recently in [4]. In the later, it was proved that $H_0(n,x)$ satisfies the homogeneous second order ordinary differential equation

$$DH_0 := A_2(x)\frac{dH_0^2}{dx^2} + A_1(x)\frac{dH_0}{dx} + A_0(x)H_0 = 0, \tag{10}$$

with

$$\begin{array}{rcl} A_2(x) &=& (x-3)(2x-3)(5x-3), \\ A_1(x) &=& -2(10nx^2 - 10x^2 - 27nx + 36x - 9n - 45), \\ A_0(x) &=& n(10nx - 3n - 10x + 21). \end{array}$$

Instead of evaluating $H_0(n,x)$ as a determinant, we shall construct it as the exact solution of equation (10). First we obtain the generating polynomials associated with the operator D: For all $k = 0, 1, 2, ...$

$$\begin{array}{rcl} P_k &:=& Dx^k = A_2(x)[k(k-1)x^{k-2}] + A_1(x)[kx^{k-1}] + A_0(x)x^k, \\ &=& 10(k-n)(k-n+1)x^{k+1} - (k-n)(51k-3n+21)x^k + \\ && 18k(n+1+4k)x^{k-1} - 27k(k-1)x^{k-2}. \end{array}$$

It can be easily seen that $\deg[P_k] = \begin{cases} n-1 & \text{if } k=n \text{ and } k=n-1, \\ k+1 & \text{otherwise.} \end{cases}$

From [8], the canonical polynomials are obtained by the recursion

$$Q_{k+1} \quad = \quad \gamma_{kn}[x^k + (k-n)(51k-3n+21)Q_k -$$

$$18k(n+1+4k)Q_{k-1} + 27k(k-1)Q_{k-2}],$$

where $\gamma_{kn} := \frac{1}{10(k-n)(k-n+1)}$. It is clear that for any n, Q_0, Q_n and Q_{n-1} are not defined by this recursion. In particular, for $k = 0, 1, 2, ..., n$, the expression of Q_k involves Q_0 and can be written as

$$Q_k(x) = q_k(x) + r_k Q_0, \tag{11}$$

where $q_0 = 0$ and $\{q_k(x)\}_{k=1}^{n-2}$ are polynomials generated by

$$\begin{aligned} q_{k+1} &= \gamma_{kn}[x^k + (k-n)(51k - 3n + 21)q_k - \\ &\quad 18k(n+1+4k)q_{k-1} + 27k(k-1)q_{k-2}], \end{aligned} \tag{12}$$

and where $r_0 = 1$ and $\{r_k\}_{k=1}^{n-2}$ are real numbers generated by

$$\begin{aligned} r_{k+1} &= \gamma_{kn}[(k-n)(51k - 3n + 21)r_k - \\ &\quad 18k(n+1+4k)r_{k-1} + 27k(k-1)r_{k-2}]. \end{aligned} \tag{13}$$

A recursive representation for $H_0(n, x)$ is now given:

Theorem 6. *Suppose that q_n and r_n are as given in (12) and (13). Then $H_0(n, x) = \alpha_n U_{n-1}(x) - \alpha_{n-1} U_n(x)$, where*

$$\begin{aligned} U_{n-1} &= x^{n-1} - (48n-30)q_{n-1} - 18(n-1)(5n-3)q_{n-2} + 27(n-1)(n-2)q_{n-3}, \\ U_n &= x^n - 18n(5n+1)q_{n-1} + 27n(n-1)q_{n-2}, \\ \alpha_{n-1} &= (48n-30)r_{n-1} + 18(n-1)(5n-3)r_{n-2} - 27(n-1)(n-2)r_{n-3}, \\ \alpha_n &= 18n(5n+1)r_{n-1} - 27n(n-1)r_{n-2}. \end{aligned}$$

The proof of this theorem is based on the following lemma.

Lemma 7. *Under the notation of Theorem 6, the following identities hold true: (i) $D[U_{n-1}] = \alpha_{n-1}$, (ii) $D[U_n] = \alpha_n$.*

Proof. We have

$$\begin{aligned} Dx^{n-1} &= (48n-30)x^{n-1} + 18(n-1)(5n-3)x^{n-2} - 27(n-1)(n-2)x^{n-3} \\ &= (48n-30)DQ_{n-1} + 18(n-1)(5n-3)DQ_{n-2} - 27(n-1)(n-2)DQ_{n-3}. \end{aligned}$$

Due to the linearity of D, the latter implies that:

$$D[x^{n-1} - (48n-30)Q_{n-1} - 18(n-1)(5n-3)Q_{n-2} + 27(n-1)(n-2)Q_{n-3}] = 0.$$

Now, using expression (11), we find that

$$D[x^{n-1} - (48n-30)(q_{n-1} + r_{n-1}Q_0) - 18(n-1)(5n-3)(q_{n-2} + r_{n-2}Q_0) + \\ + \ 27(n-1)(n-2)(q_{n-3} + r_{n-3}Q_0)] = 0.$$

Collecting the Q_0's we get

$$D[x^{n-1} - (48n-30)q_{n-1} - 18(n-1)(5n-3)q_{n-2} + 27(n-1)(n-2)q_{n-3}] = \\ D[\{(48n-30)r_{n-1} + 18(n-1)(5n-3)r_{n-2} - 27(n-1)(n-2)r_{n-3}\}Q_0].$$

Since $DQ_0 = 1$,

$$D[x^{n-1} + (48n-30)q_{n-1} - 18(n-1)(5n-3)q_{n-2} + 27(n-1)(n-2)q_{n-3}] = \\ (48n-30)r_{n-1} + 18(n-1)(5n-3)r_{n-2} - 27(n-1)(n-2)r_{n-3} \equiv \alpha_{n-1},$$

which gives identity (i). To prove (ii) we use the same arguments:

$$\begin{aligned} &Dx^n = 18n(5n+1)x^{n-1} - 27n(n-1)x^{n-2}, \\ &D[x^n - 18n(5n+1)Q_{n-1} + 27n(n-1)Q_{n-2}] = 0, \\ &D[x^n - 18n(5n+1)(q_{n-1} + r_{n-1}Q_0) + 27n(n-1)(q_{n-2} + r_{n-2}Q_0)] = 0, \\ &D[x^n - 18n(5n+1)q_{n-1} + 27n(n-1)q_{n-2}] \\ &\quad = D[\{-18n(5n+1)r_{n-1} + 27n(n-1)r_{n-2}\}Q_0] \\ &\quad = 18n(5n+1)r_{n-1} - 27n(n-1)r_{n-2} \equiv \alpha_n. \end{aligned}$$ □

Proof of Theorem 6. Multiplying (i) and (ii) of Lemma 7 by α_n and α_{n-1} respectively, then the difference $\alpha_n(i) - \alpha_{n-1}$(ii) gives

$$\alpha_n D[\{x^{n-1} + (48n-30)q_{n-1} - 18(n-1)(5n-3)q_{n-2} + 27(n-1)(n-2)q_{n-3}\}] \\ - \alpha_{n-1} D[\{x^n - 18n(5n+1)q_{n-1} + 27n(n-1)q_{n-2}\}] = 0,$$

$$D[\alpha_n\{x^{n-1} + (48n-30)q_{n-1} - 18(n-1)(5n-3)q_{n-2} + 27(n-1)(n-2)q_{n-3}\} \\ - \alpha_{n-1}\{x^n - 18n(5n+1)q_{n-1} + 27n(n-1)q_{n-2}\}] = 0.$$ □

Finally, we illustrate the algorithm given in Theorem 6 when $n = 4$.

Example 3. Let $n = 4$. Then (12) and (13) become

$$\begin{aligned} q_{k+1} &= \frac{1}{10(k-4)(k-3)}[x^k - (k-4)(51k-9)q_k - 18k(5+4k)q_{k-1} + 27k(k-1)q_{k-2}], \\ r_{k+1} &= \frac{1}{10(k-4)(k-3)}[-(k-4)(51k-9)r_k - 18k(5+4k)r_{k-1} + 27k(k-1)r_{k-2}], \end{aligned}$$

with $q_0 = 0$ and $r_0 = 1$. This gives

$$\begin{aligned} q_1(x) &= \frac{1}{120}, \quad q_2(x) = \tfrac{x}{60} - \tfrac{1}{40}, \quad q_3(x) = \frac{x^2}{20} - \frac{37x}{200} + \frac{33}{400}, \\ r_1 &= -\frac{3}{10}, \qquad r_2 = -\tfrac{9}{5}, \qquad r_3 = \frac{297}{10}, \end{aligned}$$

$$\begin{aligned} U_3(x) &= x^3 - \frac{81x^2}{10} + \frac{1467x}{100} + \frac{2187}{200}, \qquad \alpha_3 = \frac{16038}{5}, \\ U_4(x) &= x^4 - \frac{378x^2}{5} + \frac{7128x}{25} - \frac{3321}{25}, \qquad \alpha_4 = \frac{227448}{5}, \\ H_0(4,x) &= \alpha_4 U_3(x) - \alpha_3 U_4(x). \end{aligned}$$

4. Conclusion

Using the Newton Interpolation formula we characterized the linear functionals that form with the generalized canonical polynomials $\{Q_k\}_{k\in N}$ a biorthogonal system. As a result of this, each Q_k is given in a determinant form with polynomial entries. Further we showed that these Q_k's can be used to evaluate some determinants with polynomial entries and defined by ordinary differential equations such as Hankel's determinants.

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