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STRONGLY F-CLEAN RINGS

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Abstract: Let R be an associative ring with identity. R is said to be strongly f-clean ring if every element of R is the sum of an idempotent and a full element which commute. We study various properties of the strongly f-clean rings.

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1. Introduction

Let R be an associative ring with identity. An element in a ring is called clean, if it is the sum of an idempotent and a unit. An element a in a ring R is called strongly clean if $a = e + u$ where $e^2 = e \in R$ and u is a unit of R such that $eu = ue$. A ring R is called clean (resp. strongly clean) if every element of R is clean (resp. strongly clean).

The f-clean rings were introduced by Li and Feng [4]. An element $a \in R$ is said to be f-clean if it can be written as the sum of an idempotent and a full element. An element $w \in R$ is said to be a full element if there exist $s, t \in R$ such that $swt = 1$. A ring R is called f-clean if every element of R is f-clean.

In this paper, we extend f-clean elements and introduce the concept of strongly f-clean elements. We study various properties of the strongly f-clean rings.

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Definition 1.1. An element $a \in R$ is said to be strongly f-clean if $a = e+w$ where $e^2 = e \in R$, w is a full element and $ew = ew$. A ring R is called strongly f-clean if every element of R is strongly f-clean.

Obviously, invertible elements are all full elements. Therefore f-clean (resp. strongly f-clean) rings are clean (resp. strongly clean). Note that f-clean and strongly f-clean rings are the extention of "additive analogs" of unit-regular and strongly regular rings, respectively. Local rings, strongly π-regular rings, one-sided perfect rings and $T_2(R)$ for any commutative local ring R, are examples of strongly f-clean rings [1, 5].

Throughout this paper $U(R)$, $Id(R)$ and $K(R)$ always stand for the set of all units, idempotents and full elements in R, respectively.

2. Main Results

Firstly, we get some basic properties of strongly f-clean rings.

Proposition 2.1. *A direct product $R = \prod R_i$ of rings R_i is strongly f-clean if and only if the same is true for each R_i.*

Proof. Let $x = (x_i) \in R$. Each R_i is strongly f-clean if and only if for each i, $x_i = e_i + w_i$ such that $e_i \in Id(R_i)$, $s_i w_i t_i = 1$ for some $s_i, t_i \in R$ and $e_i w_i = w_i e_i$ if and only if $x = e + w$, where $e = (e_i) \in Id(R)$, $w = (w_i) \in R$ such that $(s_i)(w_i)(t_i) = (1)$ and $ew = (e_i)(w_i) = (w_i)(e_i) = we$ if and only if x is strongly f-clean. □

A routine elementary argument establishes the following result.

Proposition 2.2. *Any homomorphic image of a strongly f-clean ring is strongly f-clean.*

The following result shows that when the corner of a strongly f-clean ring is strongly f-clean.

Proposition 2.3. *Let R be a strongly f-clean ring and e be a central idempotent in R. Then eRe is also strongly f-clean.*

Proof. Let $a \in eRe$ with $a = g + f$ where $g \in Id(R)$, $f \in K(R)$ and $gf = fg$. Therefore, $a = ege + efe = ge + fe$. It is easy to show that $ge \in Id(eRe)$, $fe \in K(eRe)$ and $gefe = fege$. □

Remark. *There exists a ring R with $a \in R$ and $e \in Id(R)$ such that a is strongly f-clean in R but eae is not strongly f-clean in eRe.*

Proof. Let $R = M_2(\mathbb{Z})$, the 2×2 matrix ring over integers ring $\mathbb{Z}$. Let $a = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$, $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then $a = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is strongly f-clean in R. But eae is not strongly f-clean in $eRe = \{\begin{pmatrix} c & c \\ 0 & 0 \end{pmatrix} \mid c \in \mathbb{Z}\}$. In fact, it is easy to check that 3 is not strongly f-clean in $\mathbb{Z}$ and there exists a ring isomorphism: $eRe \simeq \mathbb{Z}$ with $\begin{pmatrix} c & c \\ 0 & 0 \end{pmatrix} \longrightarrow c$, while $eae = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}$. And so we are done. □

A ring R is called left quasi-duo (resp. right quasi-duo) if every maximal left (right) ideal of R is an ideal. A ring R is called Abelian if all idempotents are central. Next we will investigate the equivalence of strongly f-cleaness and strongly cleaness.

Proposition 2.4. *For a left quasi-duo ring or an Abelian ring R, the strongly f-cleaness and strongly cleaness are equivalent.*

Proof. It is trivial by [4]. □

Let R be a ring in which $2 \in U(R)$. Camillo and Yu [2], showed that R is clean if and only if every element of R is the sum of a unit and a square root of 1. We get analogous result for strongly f-clean rings.

Proposition 2.5. *Let $2 \in U(R)$. Then R is strongly f-clean if and only if every element of R is the sum of a full element and a square root of 1 which commute.*

Proof. This is proved in Li-Feng [4]. We sketch the proof here for the convenience of reader. If R is strongly f-clean and $a \in R$, let $\frac{1}{2}(a+1) = e+u$ where $e \in Id(R)$, $u \in K(R)$ and $eu = ue$. Then $a = 2u + (2e-1)$ is the desired decomposition. Conversely, if $2a - 1 = u + z$ with $t \in Id(R)$, $z \in K(R)$ and $uz = zu$, then $a = \frac{1}{2}u + \frac{1}{2}(z+1)$ does it. □

If R is a ring and G is a group, let RG denote the group ring.

Proposition 2.6. *Let $2 \in U(R)$ and $G = \{1, g\}$. Then RG is strongly f-clean if and only if R is strongly f-clean.*

Proof. As an image of RG, R is strongly f-clean when RG is strongly f-clean. Conversely, since $2 \in U(R)$, we have $RG \simeq R \times R$ via the map $\theta : a + bg \longrightarrow (a+b, a-b)$, [3]. Hence RG is strongly f-clean by Proposition 2.1.

Let R be a ring and let ${}_RV_R$ be an R-R-bimodule which is a general ring (possibly with no unity) in which $(vw)r = v(wr), (vr)w = v(rw)$ and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in R$. Then the ideal-extention $I(R;V)$ of

R by V is defined to be the additive abelian group $I(R;V) = R \oplus V$ with multiplication $(r,v)(s,w) = (rs, rw + vs + vw)$. Note that if S is a ring and $S = R \oplus A$, where R is a subring and $A \lhd S$, then $S \simeq I(R;A)$.

Proposition 2.7. *An ideal-extention $S = I(R;V)$ is strongly f-clean if the following conditions are satisfied:*

(a) R is strongly f-clean;

(b) if $e^2 = e \in R$ then $ev = ve$ for all $v \in V$;

(c) if $v \in V$ then $v + w + vw = 0$ for some $w \in V$.

Proof. Let $s = (r,v) \in S$ then by (a), $r = e + u$ where $e \in Id(R), u \in K(R)$ and $eu = ue$. Then $s = (e,0) + (u,v)$. Obviously, $(e,0)$ is an idempotent in S, and we will show that $(u,v) \in K(S)$. Assume that $sut = 1$. For $svt \in V$, there exists $w \in V$ such that $svt + w + wsvt = 0$ by (c), and one can check that $(s,ws)(u,v)(t,0) = 1$. Hence $(u,v) \in K(S)$ and $(e,0)(u,v) = (u,v)(e,0)$, by (b). Therefore S is strongly f-clean. □

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