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DIHEDRAL GROUPS OF ORDER 2^{m+1}

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Abstract: The aim of this paper is to determine the structure of the dihedral groups of order 2^{m+1}, where m in a natural number greater than 1, and to describe a lot of properties for these groups.

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1. Introduction

The dihedral groups play a significant rule in the Group Theory, while the dihedral groups are originally produced from the symmetries of regular polygons, which together form surfaces and planes.

The study of the theory of groups form a basis to ensure a successful achievement of a higher level. Some important applications with 3-dimensions space can be described like a lot of symmetries of polygons. Physicists associated

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with the use of groups based on the simplistic analogies in engineering study problems related to mechanics and optics, and emphasize that the tools provided by the set theory can help essentially to address some of the most difficult questions and contemporary research.

Jaume Aguadé et al, in [4], gave a new classification of the infinite dihedral groups, and they showed that a complete classification of all representations can be described by a system of numerical invariants for the dihedral group of rank 2. In [1] and [2], Conrad Keith noted that the size of D_n is at most $2n$ and every rotation in the dihedral group is conjugate to its inverse. In [7] Miller proved that it is the only dihedral group which does not admit any outer automorphisms.

We consider this class of dihedral groups of order 2^{m+1} because of its various properties, that no other classes of dihedral groups may have it all. Corollary 2 gives as consequence, a collection of properties.

2. Definitions

A dihedral group is a group of rotations and reflections for a regular polygon, the dihedral group for n-polygon is denoted by D_{2n}, where the order of this group is the number of rotations and reflections for the vertices of n-polygon, That is by determining the symmetric axes (which depend on n odd or even), and then find the reflections and rotations in term of each symmetric axis. The number of distinct rotations is n which is also the number of distinct reflections, so $|D_{2n}| = 2n$, this is why we use the notation D_{2n}. In general, let $S = \{s_0, s_1, \cdots, s_{n-1}\}$ be the set of all reflection symmetries and $R = \{r_0, r_1, \cdots, r_{n-1}\}$ be the set of all rotational symmetries both are outcomes by permutate the vertices of n-polygon, then we can give the following definition.

Definition 1. A dihedral group D_{2n} for the regular n-polygon is the set $S \cup R$ equipped with the composition operation $\circ$, given by the following relations:

$$r_i \circ r_j = r_{(i+j)\, mod\, n}\ ,\quad r_i \circ s_j = s_{(i+j)\, mod\, n}\ ,\quad s_i \circ r_s = s_{(i-j)\, mod\, n}$$

and $s_i \circ s_j = r_{(i-j)\, mod\, n}$, where the composition of symmetries is also a symmetric. Notice that $r_0 = e$ the counter clockwise rotations by $0°$ is the identity element.

Example 1. The following is the table of $D_{2(4)}$, the group of all reflections and rotations for 4-polygon:

$\circ$	e	r_1	r_2	r_3	s_0	s_1	s_2	s_3
e	e	r_1	r_2	r_3	s_0	s_3	s_2	s_1
r_1	r_1	r_2	r_3	e	s_1	s_0	s_3	s_2
r_2	r_2	r_3	e	r_1	s_2	s_1	s_0	s_3
r_3	r_3	e	r_1	r_2	s_3	s_2	s_1	s_0
s_0	s_0	s_3	s_2	s_1	e	r_1	r_3	r_2
s_1	s_1	s_0	s_3	s_2	r_1	e	r_3	r_2
s_2	s_2	s_1	s_0	s_3	r_2	r_1	e	r_3
s_3	s_3	s_2	s_1	s_0	r_3	r_2	r_1	e

$$\rightarrowtail \left[\begin{array}{c|c} R & S \\ \hline S & R \end{array}\right]$$

Table 1.

3. Dihedral Group of Order 2^{m+1}

Theorem 1. (see [1]) *The size of D_{2n} is at most $2n$.*

Proof. See [1]. □

Now we are going to explain the following problem presented in [8], page 45.

Problem. Show that the dihedral group D_{2n} is generated by two elements $a,\ b$ such that $a^n = b^2 = (ab)^2 = e$.

Answer. Let $n \geq 3$, then the rotations set could be determined by rotate the vertices of regular n-polygon counterclockwise by $2\pi/n$ each time, so r_1 is the rotation by $2\pi/n$, and r_2 is the rotating of the result rotation by $2\pi/n$, and so on, until the vertices get back to its first place, and this happen when the composite rotation process by n-times, that is $n\,(2\pi/n) = 2\pi$. So, one can generate all the rotations in term of the base rotation $r = r_1$, and then repeat r for any order $0 < i \leq n$ to get the other rotations. For our aims, let r be the rotation clockwise of n-polygon by $2\pi/n$, then the set of rotations R is $\{r^i \mid i = 1, 2, 3, \cdots, n\}$, and it can be easily shown that the order of r is n; i.e $r^n = e$, where e is the rotation by $0^\circ \equiv 2m\pi\ ,\ m \in \mathbb{N}$. To describe the reflections elements of D_{2n}, let us begin with the base reflection $s = s_0$ which is the reflection of n-polygon about the horizontal axis passing through the center of the polygon, and then all other reflections can be made by rotate this reflection counter clockwise by

$i(2\pi/n)$, $i = 1, 2, 3, \cdots, n-1$. By this way, we get $n-1$ rotations for the base reflection s, and then n reflections, this gives the set of reflections S to be $S = \{s\} \cup \{r^i \circ s \mid i = 1, 2, 3 \cdots, n-1\} = \{r^i \circ s \mid \ i = 1, 2, 3, \cdots, n\}$, it appears that s is not a power of r and so all the other reflections, it is clear that $|s| = 2$ and $(r^k s)^2 = e$. This gives the following result (We will write rs instead of $r \circ s$):

Corollary 1. *All the elements of D_{2n} can be written as a composition of rotations and reflections. So, the minimal number of generators is 2, which of the form r^k for rotations and $r^k s$ for the reflections, $0 < k \leq n$.*

Proof. The proof is straightforward from the previous descriptions of both r and s. □

In another point of view, $(r^k)^{-1} = r^{n-k}$, where the composition operation of r^k and r^{n-k} should be the identity. To show that every rotation is conjugate to its inverse; we are first looking for $e \neq a \in D_{2n}$ where $ara^{-1} = r^{-1}$, then a is a rotation or a reflection, if a is a rotation then $a = r^k$ for some integer $k < n$, and then $r^k r (r^k)^{-1} = r^{k+1-k} = r \neq r^{-1}$, then our only choice of a is $a = s$ as a reflection. This gives that $srs^{-1} = r^{-1}$, in general $sr^k s^{-1} = (r^k)^{-1} = r^{-k}$, using this description we get the following definition:

Definition 2. The dihedral group D_{2n} is the group generated by r and s, where:

$$r^n = s^2 = (r^k s)^2 = e \ \text{ for } \ k = 1, 2, \cdots, n.$$

So one can write

$$D_{2n} = \left\langle r, s \mid r^n = s^2 = (r^k s)^2 = e, \ \ k = 1, 2, \cdots, n \right\rangle.$$

As a direct consequence of all the above, the order of any reflection is 2, where $(r^k s)^{-1} = r^k s$ and $(r^k)^{-1} = r^{n-k}$.

Remark 1. Let $G = D_{2n}$, where $n = 2^m$, and m is a natural number greater than 1. Then $cl_G = 5(2)^{m-2} - 3 \sum_{k=0}^{m-3} 2^k = 3 + 2^{m-1}$

Proof. The conjugacy classes of G are $5, 7, 11, 19, 35, 67$ for $m = 2, 3, 4, 5, 6, 7$ respectively. So we construct the sequence $\{a_i\}_{i=1}^{\infty}$, where $a_1 = 5$ and $a_i =$

$2a_{i-1} - 3$. Then, with some calculus works, we modify these terms to get

$$a_i = 2^{i-1}a_1 - 3(2)^{i-2} - 3(2)^{i-3} - \cdots - 3(2)^0 = 5(2)^{i-1} - 3\sum_{k=0}^{i-2} 2^k,$$

but $m = i+1$, after replacing i, then the right side will give the required values, and the result terms are 5 for $m = 2$ and $5(2)^{m-2} - 3\sum_{k=0}^{m-3} 2^k = 3 + 2^{m-1}$ for $m \geq 3$. □

Theorem 2. *Under the same assumptions of Remark 1, the group G is nilpotent of class m.*

Proof. Let $D_{2n} = \langle r, s \mid r^n = s^2 = (r^k s)^2 = e, \ \ k = 1, 2, \cdots, n \rangle$. Recall that the lower central series of a group G is

$$G = G_1 \trianglelefteq G_2 \trianglelefteq G_3 \trianglelefteq \cdots \trianglelefteq G_k \trianglelefteq \cdots,$$

where $G_1 = G$, and $G_{k+1} = [G_k, G]$, note that $[G_k, G] = \{[x, y] \mid x \in G_k\, , \ y \in G\}$, the group G is nilpotent if $[G_k, G] = e$ for some k, and the smallest such k is the class of nilpotency.

Let $G = D_{2n}$, where $n = 2^m$, then $G_2 = [G, G] =< r^2 >$ and $|G_2| = 2^m/2 = 2^{m-1}$, Consequently:

$|G_3| = |[G_2, G]| = | < r^4 > | = 2^m/2^2 = 2^{m-2}, \cdots, |[G_k, G]| = 2^m/2^k = 2^{m-k}$, All G_i are well define for $i = 1, 2, \cdots, k$ and $|[G_m, G]| = 2^{m-m} = 1$ then $[G_m, G]$ is the trivial group, therefore, G is nilpotent of class m. □

The following corollary is a consequence result of Theorem 2.

Corollary 2. *Let G be the dihedral group of order $2n$, where $n = 2^m$, $m > 1$. Then the following hold:*

1. *The group G is solvable, and $G^{(2)} = [[G, G], [G, G]] = \{e\}$.*
2. *If $m = 1$ then G is Abelian group, and G is not Abelian group for $m \geq 2$.*
3. *G has no proper Sylow p-subgroups, and the only Sylow 2-subgroup of G is G itself.*
4. *The radical of G is also G, i.e., the largest solvable normal subgroup of G is G.*

5. $|G'| = |[G : G]| = 2^{m-1}$. *Moreover, the order of the factor group of* G *is* $|F(G)| = |G|_{G'}| = 4$.

6. $Z(G) = \{e, r^{2^{m-1}}\}$, *where* $Z(G)$ *is the center of* G.

Proof. 1. Every nilpotent group is solvable, and $G_2 = [G, G] =< r^2 >$, then

$$\begin{aligned} [[G,G],[G,G]] &= [< r^2 >, < r^2 >] \\ &= \{[x,y] \mid x = r^{2k}, \ y = r^{2m}, \\ &\quad k = 1, 2, \cdots, \tfrac{n}{2}, \ m = 1, 2, \cdots, \tfrac{n}{2}\} \\ &= \{e\}. \end{aligned}$$

2. For $m = 1$, then $D_{2n} = D_4 = \{e, r, s, rs\}$, it is easy to check that D_4 is Abelian, where $sr = srss = r^{-1}s = rs$ but for $m > 1$, then $|r| > 2$ so $rs \neq sr$, hence D_{2n} is not Abelian group for $m > 1$.

3. Let $G = D_{2n}$ where $n = 2^m$, then $|G| = 2^{m+1}$. Then $|x| = 2$ if x is reflection and $|x| = n = 2^{m+1}$ for rotations, so G is 2-group, the only prime devisor of $|G|$ is 2. So there is only one Sylow 2-subgroup of G which G.

4. This is a consequence result from item 3.

5. It is clear from the proof of Theorem 2.2.

6. The center of the dihedral group is $\{e\}$ if n is odd, or $\{e, x\}$ if n is even. For G is a dihedral group of even order, then $|Z(G)| = 2$, x is the only element which commute with all other elements, so x must be the rotation of 180°. That is, the n rotations by $2\pi/n$ gives the identical rotation 360°, and $\frac{n}{2}$ rotations gives π which is the convex situation. So, for all $y \in G$, we have, $r^{n/2}y = yr^{n/2}$. Then $r^{n/2} = r^{2^{m-1}} \in Z(G)$.

□

References

[1] Conrad Keith, *Dihedral Groups*, Retrieved from: http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/dihedral.pdf.

[2] Conrad Keith, *Dihedral Groups II* Retrieved from: http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/dihedral2.pdf.

[3] David S. Dummit, Richard M. Foote, *Abstract Algebra*, 3-rd Ed., John Wiley and Sons (2004), ISBN 0-471-43334-9.

[4] Jaume Aguadé, Carles Broto, and Laia Saumell, Rank two integral representations of the infinite dihedral group, *Communications in Algebra*, **35** (2007), 1539-1551.

[5] I. Martin Issacs, *Finite Group Theory*, American Mathematical Society (2008), ISBN 978-0-8218-4344-4.

[6] Marlos Viana, Vasudevan Lakhsminarayanan, Dihedral representations and statistical geometric optics, II: Elementary optical instruments, *Journal of Modern Optics*, **54**, No. 4 (2007), 473-485.

[7] G.A. Miller, Automorphisms of the dihedral groups, *Proc. N.A.S.*, **28** (1942), 368-371.

[8] Morton Hamermesh, *Group Theory and its Application to Physical Problems*, Addison Wesley (1989), ISBN 0-486-66181-4.