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A NOTE ON SOME PROPERTIES OF DOUBLE TOPOLOGICAL OPERATIONS AND NEARLY OPEN SETS

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Abstract: The aim of this paper is to study some topological double operations $\left\{(A^\circ)^\circ, (A^\circ)^E, (A^\circ)^b, \left(A^E\right)^\circ, \cdots\right\}$ and to obtain the interrelationships between these operations. We also study and investigate new properties of nearly open sets and introduce some conditions to strengthen semi pre-open (pre-closed) sets to be open (closed).

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1. Introduction

This paper consists of preliminaries (Section 2) and two basic sections. Section 3 studies the relationships between double operations, and by using these relationships we prove that $E(P^\circ)^s, E(P^E)^s$, and $E(P^b)^s$ form a partition of whole space X (or fewer when one or more of these elements of each set is nonempty) in spite of $E(P)^s$ form a partition of X. The concepts of semiopen (semiclosed), pre-open (pre-closed), and regular open (regular closed) were introduced and studied, respectively in [3],[4],[5], and [6].

In Section 4 we rewrite Definition 2.3 using the properties of double operations, and provide also some theorems and remarks on this subject.

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2. Preliminaries

Throughout this paper (X, τ) denotes the topological space, $\overline{A}$ is the closure of A, A° is the interior of A, A^E is the exterior of A, A^b is the boundary of A,and A^C is the complement of A.

Definition 1. ([1]) Let X be a nonempty set. By a partition p of X we mean a set of nonempty subsets of X such that:

1. If $A, B \in P$ and $A \neq B$, then $A \cap B = \emptyset$.
2. $\bigcup_{c \in P} c = X$.

Definition 2. ([2]) The partition topology is a topology that can be induced on any set X by partitioning X into disjoint subsets P, these subsets form the basis for the topology.

Definition 3. A subset U of a topological space (X, τ) is said to be:

1. Semiopen set if $U \subseteq \overline{U^\circ}$.
2. Semiclosed set if $\overline{U}^\circ \subseteq U$.
3. Preopen set if $U \subseteq \overline{U}^\circ$.
4. Preclosed set if $\overline{U^\circ} \subseteq U$.
5. Regular open set if $U = \overline{U}^\circ$
6. Regular closed if $U = \overline{U^\circ}$.
7. α-open set if $U \subseteq \left(\overline{U^\circ}\right)^\circ$.
8. α-closed set if $\overline{\overline{U}^\circ} \subseteq U$.

Theorem 1. *([9]) Let (X, τ) be a topological space and let $A \subseteq X$. Then*

1. $A^\circ = \left(\overline{(A^c)}\right)^c$.
2. $\overline{A} = ((A^c)^\circ)^c$.

Remark 1. Using the same assumptions of the previous theorem it follows:

$$A^E = (A^c)^\circ = \left(\overline{A}\right)^c. \tag{1}$$

Theorem 2. *([8]) Let (X, τ) be a topological space and let $A \subseteq X$. Then:*

1. $A^b = \overline{A} \cap \overline{(A)^c}$
2. $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$
3. $(A \cap B)^\circ = A^\circ \cap B^\circ$.

Proposition 1. *Let (X, τ) be a topological space and let $A \subseteq X$. Then:*

1. *A is clopen set if and only if $A^b = \emptyset$*
2. $A^b = (A^c)^b$
3. $\left(\overline{A}\right)^b \subseteq A^b$.
4. $(A \cap B)^b \subseteq A^b \cup B^b$.

Proof. 1. If A is clopen, then $A^b = \overline{A} \cap \overline{(A)^c} = A \cap A^c = \emptyset$. Conversely, if $A^b = \overline{A} \cap \overline{(A)^c} = \emptyset$, then $\overline{A} \subseteq \left(\overline{(A)^c}\right)^c = A^\circ$, using Remark 1 it follows $A \subseteq \overline{A} \subseteq A^\circ \subseteq A$, which implies that A is clopen.

2. It is clear, from Theorem 2, Part 1.

3. $\left(\overline{A}\right)^b = \overline{A} \cap \overline{\left(\left(\overline{A}\right)^c\right)} \subseteq \overline{A} \cap \overline{(A)^c} = A^b$.

4.
$$\begin{aligned} (A \cap B)^b &= (\overline{A \cap B}) \cap \overline{(A \cap B)^c} \\ &= (\overline{A \cap B}) \cap (\overline{A^c} \cap \overline{B^c}) \\ &\subseteq (\overline{A} \cap \overline{B}) \cap (\overline{A^c} \cap \overline{B^c}) \\ &= \left(\overline{A} \cap \overline{B} \cap \overline{A^c}\right) \cup \left(\overline{A} \cap \overline{B} \cap \overline{B^c}\right) \\ &= \left(\overline{A^b} \cap \overline{B}\right) \cup \left(\overline{A} \cap \overline{B^b}\right) \subseteq A^b \cup B^b \end{aligned}$$

$\square$

Theorem 3. *([8]) Let (X, τ) be a topological space and let $A \subseteq X$. Then:*

1. *A is open if and only if $A^b \cap A = \emptyset$.*
2. *A is closed if and only if $A^b \subseteq A$.*

3. Double topological operations

Definition 4. Let $P = \left\{A^{\circ}, A^{E}, A^{b}\right\}$ be the partition of the space X, and $E(P)^{s}$ denote the elements of P. Define:

$$P^{\circ} = \left\{(A^{\circ})^{\circ}, (A^{\circ})^{E}, (A^{\circ})^{b}\right\},$$

$$P^{E} = \left\{\left(A^{E}\right)^{\circ}, \left(A^{E}\right)^{E}, \left(A^{E}\right)^{b}\right\},$$

$$P^{b} = \left\{\left(A^{b}\right)^{\circ}, \left(A^{b}\right)^{E}, \left(A^{b}\right)^{b}\right\},$$

with elements of P°, P^{E}, and P^{b}, denoted by $E(P^{\circ})^{s}$, $E(P^{E})^{s}$, and $E(P^{b})^{s}$ respectively.

In this section we study the relationships between double operations. Using these relationships we prove that $E(P^{\circ})^{s}$, $E(P^{E})^{s}$, and $E(P^{b})^{s}$ form a partition of the space X (or fewer, when one or more element of each set is nonempty). Besides, $E(P)^{s}$ forms a partition of X as follows:

Lemma 1. *Let (X, τ) be a topological space and let $A \subseteq X$. Then X can be expressed as disjoint union of elements of P, i.e $X = A^{\circ} \cup A^{E} \cup A^{b}$, that is $E(P)^{s}$ form a partition of X.*

Proof. We have $X = A^{b} \cup \left(A^{b}\right)^{c}$, but

$$A^{b} = \overline{A} \cap \overline{A^{c}} = \left(\left(\overline{A}\right)^{c} \cup \left(\overline{A^{c}}\right)^{c}\right)^{c} = \left(A^{E} \cup A^{\circ}\right)^{c},$$

by Theorem 1 and Remark 1, the complement of both sides gives $\left(A^{b}\right)^{c} = A^{E} \cup A^{\circ}$. Clearly $X = A^{\circ} \cup A^{E} \cup A^{b}$. To show that these elements are mutually disjoint, we have:

$$A^{\circ} \cap A^{E} = A^{\circ} \cap (A^{c})^{\circ} = (A \cap A^{c})^{\circ} = \emptyset^{\circ} = \emptyset,$$

$$A^{\circ} \cap A^{b} = \left(\overline{A^{c}}\right)^{c} \cap \overline{A} \cap \overline{(A^{c})} = \overline{A} \cap \left(\left(\overline{A^{c}}\right)^{c} \cap \overline{(A^{c})}\right) = \overline{A} \cap \emptyset = \emptyset,$$

with $A^{E} \cap A^{b} = \left(\overline{A}\right)^{c} \cap \overline{A} \cap \overline{(A^{c})} = \overline{A^{c}} \cap \left(\left(\overline{A}\right)^{c} \cap \overline{A}\right) = \overline{A^{c}} \cap \emptyset = \emptyset$. Hence $E(P)^{s}$ form a partition of X. $\square$

Lemma 2. $((A^{\circ})^{c})^{\circ} = \left(\overline{(A^{\circ})}\right)^{c} = \left(\overline{(A)^{c}}\right)^{\circ}$.

Proof. Since $(A^\circ)^c = \overline{(A^c)}$, then $((A^\circ)^c)^\circ = \left(\overline{(A)^c}\right)^\circ$, and as

$\overline{A} = ((A^c)^\circ)^c$, then $\overline{A^\circ} = \left(((A^\circ)^c)^\circ\right)^c$, it follows $\left(\overline{(A)^\circ}\right)^c = ((A^\circ)^c)^\circ$.

∎

Theorem 4. *Let (X, τ) be a topological space and let $A \subseteq X$. Then:*

	A°	A^E	A^b
A°	$(A^\circ)^\circ = A^\circ$	$(A^\circ)^E = \left(\overline{(A)^c}\right)^\circ \subseteq (A^\circ)^c$	$(A^\circ)^b = \overline{(A)^\circ} \cap \overline{(A)^c} \subseteq A^b$
A^E	$\left(A^E\right)^\circ = A^E$	$\left(A^E\right)^E = \left(\overline{A}\right)^\circ$	$\left(A^E\right)^b = \left(\overline{A}\right)^b \subseteq A^b$
A^b	$\left(A^b\right)^\circ = \left(\overline{A}\right)^\circ \cap ((A^\circ)^c)^\circ = \left(\overline{A}\right)^\circ \cap \left(\overline{(A^\circ)}\right)^c = \left(\overline{A}\right)^\circ \cap \left(\overline{(A)^c}\right)^\circ \subseteq A^b$	$\left(A^b\right)^E = \left(A^\circ \cup A^E\right)^\circ = \left(A^b\right)^c = A^\circ \cup A^E$	$\left(A^b\right)^b = \overline{(A^\circ \cup A^E)} \cap A^b \subseteq A^b$

Proof. 1. $(A^\circ)^\circ = A^\circ$, since A° is an open set.

2. $\left(A^E\right)^\circ = ((A^c)^\circ)^\circ = (A^c)^\circ = A^E$.

3. $(A^\circ)^E = ((A^\circ)^c)^\circ = \left(\overline{(A)^c}\right)^\circ$

$$\subseteq \overline{(A)^c} = (A^\circ)^c.$$

4. $\left(A^E\right)^E = \left(\left(A^E\right)^c\right)^\circ = \left(\left(\left(\overline{A}\right)^c\right)^c\right)^\circ = \left(\overline{A}\right)^\circ$.

5.

$$\begin{aligned} \left(A^b\right)^\circ &= \left(\overline{A} \cap \overline{(A)^c}\right)^\circ \\ &= \left(\overline{A}\right)^\circ \cap \left(\overline{A^c}\right)^\circ \subseteq \overline{A} \cap \left(\overline{A^c}\right) = A^b \\ &= \left(\overline{A}\right)^\circ \cap ((A^\circ)^c)^\circ, \text{ by Lemma 3,} \\ &= \left(\overline{A}\right)^\circ \cap \left(\overline{A^\circ}\right)^c, \text{ by Lemma 3.} \end{aligned}$$

6. $(A^\circ)^b = \overline{A^\circ} \cap \overline{(A^\circ)^c} = \overline{A^\circ} \cap (A^\circ)^c$, but $\overline{A^\circ} \subseteq \overline{A}$, $(A^\circ)^c = \overline{A^c}$, then $(A^\circ)^b \subseteq \overline{A} \cap \overline{(A)^c} = A^b$.

7. $\left(A^E\right)^b = \left(\left(\overline{A}\right)^c\right)^b = \left(\overline{A}\right)^b \subseteq A^b$.

8.

$$\begin{aligned} \left(A^b\right)^E &= \left(\left(A^b\right)^c\right)^\circ = \left(\left(\overline{A} \cap \overline{A^c}\right)^c\right)^\circ \\ &= \left(\left(\overline{A}\right)^c \cup \left(\overline{A^c}\right)^c\right)^\circ = \left(A^\circ \cup A^E\right)^\circ \supseteq A^\circ \cup A^E = \left(A^b\right)^c . \end{aligned}$$

But we have:

$$\left(A^\circ \cup A^E\right)^\circ \subseteq A^\circ \cup A^E, \text{ then } \left(A^b\right)^E = \left(A^E \cup A^\circ\right)^\circ = A^E \cup A^\circ = \left(A^b\right)^c.$$

9. $\left(A^b\right)^b = \left(\overline{A} \cap \overline{A^c}\right)^b \subseteq \left(\overline{A}\right)^b \cup \left(\overline{A^c}\right)^b \subseteq A^b \cup (A^c)^b = A^b \cup A^b = A^b$. Hence $\left(A^b\right)^b \subseteq A^b$, and clearly $\left(A^b\right)^b = \overline{A^b} \cap \overline{(A^b)^c} = A^b \cap \overline{(A^E \cup A^\circ)}$.

□

Remark 2. We have $\left(A^E\right)^\circ \subseteq (A^\circ)^E$ but the converse is not true.

Since $A^\circ \subseteq A$ it follows that $\left(A^E\right)^\circ \subseteq (A^\circ)^E$. Or by

$$A^E = \left(\overline{A}\right)^c \subseteq A^c \subseteq (A^\circ)^c, \text{ then } \left(A^E\right)^\circ = A^E \subseteq ((A^\circ)^c)^\circ = (A^\circ)^E.$$

The following example shows that the converse is not true.

Example 1. Consider $A = [1, 2] \subseteq X = \mathbb{R}$ with co-countable topology, then $A^\circ = A^E = \emptyset$, but $(A^\circ)^E = (\emptyset)^E = X$

$$\Rightarrow (A^\circ)^E \neq A^\circ,\ (A^\circ)^E \neq A^\circ$$
$$\text{and } (A^\circ)^E \not\subseteq \left(A^E\right)^\circ.$$

Remark 3. $(A^\circ)^E$ needs not to be the same as A^E or A°, in spite if A^E is an open set.

Remark 4.

$$A^\circ \subseteq \left(A^E\right)^E \subseteq \overline{A}. \tag{2}$$

Since

$$\left(\overline{A}\right)^\circ = \left(A^E\right)^E \subseteq \overline{A} \tag{3}$$

and

$$A^E \subseteq A^c, \text{ then } (A^c)^E = A^\circ \subseteq \left(A^E\right)^E. \tag{4}$$

Now by combining (3) and (4), we get

Remark 5. $\left(A^b\right)^\circ = \left(A^E\right)^E \cap (A^\circ)^E$.

This follows obviously from part (3) and part (4). Also we have

$$\left(A^b\right)^\circ = \left(A^E\right)^E \cap (A^\circ)^E = \left(A^E \cup A^\circ\right)^E = \left(\left(A^b\right)^c\right)^E.$$

Corollary 1. *From Theorem 1 we have the following results:*

1.

$$\left(A^b\right)^\circ = \left(A^E\right)^E \cap (A^\circ)^E. \quad (5)$$

Remark 4 implies that $\left(A^b\right)^\circ \subseteq \left(A^E\right)^E, \left(A^b\right)^\circ \subseteq (A^\circ)^E$.

2. $\left(A^E\right)^\circ \subseteq \left(A^b\right)^E$ *is obvious from Theorem 1, parts 2 and 8.*

3. Since $\left(A^b\right)^\circ = \left(\overline{A}\right)^\circ \cap ((A^\circ)^c)^\circ$ *and* $(A^\circ)^b = \overline{A^\circ} \cap (A^\circ)^c$ *which implies that* $\left(A^b\right)^\circ \subseteq \left(\overline{A}\right)^\circ = \left(A^E\right)^E$ *and* $(A^\circ)^b \subseteq \overline{A^\circ}$.

Note 1. $\overline{A^\circ} = \left(((A^\circ)^c)^\circ\right)^c$ and $\left(\overline{A}\right)^\circ = \left(\overline{\left(\left(\overline{A}\right)^c\right)}\right)^c$.

Note 2. It is easy to show that $(A^\circ)^b \subseteq \overline{A^\circ} = \left(((A^\circ)^c)^\circ\right)^c = \left((A^\circ)^E\right)^c$.

Theorem 5. *1. Let (X, τ) be a topological space and let $A \subseteq X$. Then X can be expressed as disjoint union of elements of P° (or simply $X = (A^\circ)^\circ \cup (A^\circ)^E \cup (A^\circ)^b$), that is $E(P^\circ)^s$ form a partition of X.*

2. X can be expressed as disjoint union of elements of P^E, or simply $X = \left(A^E\right)^\circ \cup \left(A^E\right)^E \cup \left(A^E\right)^b$), or $E(P^E)^s$ form a partition of X.

3. X can be expressed as disjoint union of elements of P^b, or simply $X = \left(A^b\right)^\circ \cup \left(A^b\right)^E \cup \left(A^b\right)^b$, or $E(P^b)^s$ form a partition of X.

Proof. Using Lemma 1, we prove the above theorem in an alternative method, that is, by using the properties of double operations as follows:

1. To prove that $E(P^\circ)^s$ form a partition of the space X (or fewer when one element or more is nonempty). We have

$$\begin{aligned} X &= (A^\circ)^b \cup \left((A^\circ)^b\right)^c = (A^\circ)^b \cup \left(\overline{A^\circ} \cap \overline{A^c}\right)^c \\ &= (A^\circ)^b \cup \left(\left(\overline{A^\circ}\right)^c \cup \left(\overline{A^c}\right)^c\right) = (A^\circ)^b \cup \left((A^\circ)^E \cup A^\circ\right), \\ &= (A^\circ)^b \cup (A^\circ)^E \cup (A^\circ)^\circ \end{aligned}$$

where $E(P^{\circ})^s$ are mutually disjoint for:

$$(A^{\circ})^{\circ} \cap (A^{\circ})^{E} = A^{\circ} \cap \left(\overline{A^c}\right)^{\circ} \subseteq A^{\circ} \cap \left(\overline{A^c}\right) = A^{\circ} \cap (A^{\circ})^{c} = \emptyset.$$

And since $A^{\circ} \subseteq \overline{A^{\circ}}$, $\overline{A^c} = (A^{\circ})^c]$, then

$$(A^{\circ})^{\circ} \cap (A^{\circ})^{b} = A^{\circ} \cap \left(\overline{A^{\circ}}\right) \cap \left(\overline{A^c}\right) \subseteq A^{\circ} \cap (A^{\circ})^{c} = \emptyset,$$

$$(A^{\circ})^{E} \cap (A^{\circ})^{b} = \left(\overline{A^c}\right)^{\circ} \cap \left(\overline{A^{\circ}}\right) \cap \left(\overline{A^c}\right) \subseteq \left(\overline{A^c}\right)^{\circ} \cap \left(\left(\overline{A^c}\right)^{\circ}\right)^{c} = \emptyset.$$

Or using Corollary 1, part (4), it shows that $E(P^{\circ})^s$ form a partition of X.

2. To prove that $E(P^{E})^s$ form a partition of X:

$$\begin{aligned} X &= \left(A^{E}\right)^{b} \cup \left(\left(A^{E}\right)^{b}\right)^{c} = \left(A^{E}\right)^{b} \cup \left(\overline{A} \cap \overline{\left(\overline{A}\right)^{c}}\right)^{c} \\ &= \left(A^{E}\right)^{b} \cup \left(\left(\overline{A}\right)\right)^{c} \cup \left(\left(\overline{\left(\overline{A}\right)^{c}}\right)^{c}\right) = \left(A^{E}\right)^{b} \cup \left(A^{E} \cup \left(\overline{A}\right)^{\circ}\right) \\ &= \left(A^{E}\right)^{b} \cup \left(A^{E}\right)^{\circ} \cup \left(A^{E}\right)^{E}. \end{aligned}$$

Also we have the following:

$$\left(A^{E}\right)^{\circ} \cap \left(A^{E}\right)^{E} = \left(\overline{A}\right)^{c} \cap \left(\overline{A}\right)^{\circ} \subseteq \overline{\left(\overline{A}\right)^{c}} \cap \left(\overline{\left(\overline{A}\right)^{c}}\right)^{c} = \emptyset,$$

$$\left(A^{E}\right)^{\circ} \cap \left(A^{E}\right)^{b} = \left(\overline{A}\right)^{c} \cap \overline{\left(\overline{A}\right)^{c}} \cap \overline{A} \subseteq \left(\overline{A}\right)^{c} \cap \overline{A} = \emptyset,$$

$$\left(A^{E}\right)^{E} \cap \left(A^{E}\right)^{b} = \left(\overline{A}\right)^{\circ} \cap \overline{\left(\overline{A}\right)^{c}} \cap \overline{A} = \left(\overline{\left(\left(\overline{A}\right)^{c}\right)}\right)^{c} \cap \overline{\left(\overline{A}\right)^{c}} \cap \overline{A} = \emptyset \cap \overline{A} = \emptyset.$$

Hence $E(P^{E})^s$ form a partition of X.

3. We have the following:

$$\begin{aligned} X &= \left(A^{b}\right)^{b} \cup \left(\left(A^{b}\right)^{b}\right)^{c} = \left(A^{b}\right)^{b} \cup \left(A^{b} \cap \overline{\left(A^{E} \cup A^{\circ}\right)}\right)^{c} \\ &= \left(A^{b}\right)^{b} \cup \left(\left(A^{b}\right)^{c} \cup \left(A^{E} \cup A^{\circ}\right)^{E}\right) \\ &= \left(A^{b}\right)^{b} \cup \left(\left(A^{b}\right)^{c} \cup \left(\left(A^{E}\right)^{E} \cap (A^{\circ})^{E}\right)\right) \\ &= \left(A^{b}\right)^{b} \cup \left(A^{b}\right)^{E} \cup \left(A^{b}\right)^{\circ}. \end{aligned}$$

Now to show that $E(P^{b})^s$ are mutually disjoint, we have:

$$\left(A^{b}\right)^{\circ} \cap \left(A^{b}\right)^{E} \subseteq A^{b} \cap \left(A^{b}\right)^{c} = \emptyset$$

$$\begin{aligned}
\left(A^{b}\right)^{\circ}\cap\left(A^{b}\right)^{b} &= \left(\overline{A}\right)^{\circ}\cap\left(\overline{A^{c}}\right)^{\circ}\cap\left(A^{b}\cap\overline{\left(A^{E}\cup A^{\circ}\right)}\right)\\
&= \left(\overline{A}\right)^{\circ}\cap\left(\overline{A^{c}}\right)^{\circ}\cap\left(A^{b}\cap\left(\overline{A^{E}}\cup\overline{A^{\circ}}\right)\right)\\
&= \left(\overline{A}\right)^{\circ}\cap\left(\overline{A^{c}}\right)^{\circ}\cap\left(A^{b}\cap\left(\overline{\left(\overline{A^{c}}\right)^{c}}\cup\overline{\left(\overline{A}\right)^{c}}\right)\right)\\
&\subseteq \left(\overline{A}\right)^{\circ}\cap\left(\overline{A^{c}}\right)^{\circ}\cap\left(\left(\overline{\left(\overline{A^{c}}\right)^{c}}\cup\overline{\left(\overline{A}\right)^{c}}\right)\right)\\
&= \left(\overline{A}\right)^{\circ}\cap\left(\overline{A^{c}}\right)^{\circ}\cap\left(\left(\left(\left(\overline{A^{c}}\right)^{\circ}\right)^{c}\cup\left(\left(\overline{A}\right)^{\circ}\right)^{c}\right)\right)\\
&\subseteq \left(\left(\overline{A}\right)^{\circ}\cap\left(\overline{A^{c}}\right)^{\circ}\cap\left(\left(\overline{A^{c}}\right)^{\circ}\right)^{c}\right)\\
&\quad\cup\left(\left(\overline{A}\right)^{\circ}\cap\left(\overline{A^{c}}\right)^{\circ}\cap\left(\left(\overline{A}\right)^{\circ}\right)^{c}\right)\\
&\subseteq \left(\left(\overline{A}\right)^{\circ}\cap\emptyset\right)\cup\left(\left(\overline{A^{c}}\right)^{\circ}\cap\emptyset\right)=\emptyset.
\end{aligned}$$

$$\left(A^{b}\right)^{E}\cap\left(A^{b}\right)^{b}=\left(A^{b}\right)^{c}\cap\left(A^{b}\cap\overline{\left(A^{b}\right)^{c}}\right)=\emptyset\cap\overline{\left(A^{b}\right)^{c}}=\emptyset.$$

Hence $E(P^{b})^{s}$ form a partition of X.

□

4. Semiopen, preopen, and α-open sets

In this section we redefine Definition 3 by means of the properties of double operations, also we provide some theorems and remarks on this subject.

Definition 5. A subset U of a topological space (X,τ) is said to be:

1. Semiopen set if $U\subseteq\left(U^{\circ E}\right)^{c}$.
2. Semiclosed if $U^{EE}\subseteq U$.
3. Preopen set if $U\subseteq U^{EE}$
4. Preclosed if $\left(U^{\circ E}\right)^{c}\subseteq U$.
5. Regular open if $U=U^{EE}$.
6. Regular closed if $U=\left(U^{\circ E}\right)^{c}$.
7. α-open set if $U\subseteq\left(U^{\circ}\right)^{EE}$.
8. α-closed set if $\overline{U^{EE}}\subseteq U$.

Remark 6. 1. If A is semiclosed set, then $\left(A^b\right)^\circ \subseteq A$.

2. If A a preclosed set, then $(A^\circ)^b \subseteq A$.

Proof. It is obvious by Corollary 1. □

Theorem 6. *Let (X, τ) be a topological space, and let $A \subseteq X$. If A is closed set subset of X, then:*

1. $A^{EE} = A^\circ$.
2. $\left(A^E\right)^b = A^b$, *therefore* $\left(A^E\right)^b \subseteq A$.
3. $\left(A^E\right)^\circ = A^c$
4. $\left(A^b\right)^E = A^{EE} \cup \left(A^E\right)^\circ$.

Proof. 1. $A^{EE} = \left(\overline{A}\right)^\circ$, but A is closed set, then $\overline{A} = A$. Thus $A^{EE} = A^\circ$.

2. $\left(A^E\right)^b = \overline{A}^b = A^b$, since A closed implies that $A^b \subseteq A$, thus $\left(A^E\right)^b = A^b \subseteq A$.

3. $\left(A^E\right)^\circ = A^E = \left(\overline{A}\right)^c = A^c$.

4. $\left(A^b\right)^E = A^\circ \cup A^E = A^\circ \cup \left(\overline{A}\right)^c = A^\circ \cup A^c = A^{EE} \cup \left(A^E\right)^\circ$.

□

Theorem 7. *Let $A \subseteq X$ be any open subset of the space X, then A contains none of its boundary interior points.*

Proof. et A be an open subset of X, we want to prove that $(A^\circ)^b \cap A = \emptyset$. Since $(A^\circ)^b \cap A = \overline{A^\circ} \cap \overline{A^c} \cap A = \overline{A} \cap \overline{A^c} \cap A$, so $(A^\circ)^b \cap A = A^b \cap A = \emptyset$. □

Remark 7. The converse of above theorem is not true, as shown in the next example.

Example 2. Consider $X = \{1, 2, 3, 4\}$, with

$$\tau = \{\emptyset, X, \{1, 2\}, \{3\}, \{1, 2, 3\}\}.$$

Take $A = \{1,4\}$, then $A^\circ = \emptyset$, and $(A^\circ)^b = (\emptyset)^b = \emptyset \subseteq A$, but A is not open set.

Theorem 8. *Let A be a semiopen set and $(A^\circ)^b \cap A = \emptyset$, then A is open set.*

Proof. If $A = \emptyset$, then A is open set, suppose that $A \neq \emptyset$ which implies that there exists $x \in A \Rightarrow x \notin (A^\circ)^b = \overline{(A^\circ)} \cap \overline{A^c}$, thus either $x \notin \overline{(A^\circ)}$ or $x \notin \overline{(A^c)}$. If $x \notin \overline{(A^\circ)} \Rightarrow x \notin A$.

Thus $x \notin \overline{(A^c)}$, then $x \in \left(\overline{(A^c)}\right)^c = A^\circ$. Hence A is open set. □

Corollary 2. *A is open set if and only if A is semiopen set and $(A^\circ)^b \cap A = \emptyset$.*

Proof. It follows by Theorem 2 and Theorem 3. It is clear that every open set is semiopen set. □

Theorem 9. *Let A be an open set, then $(A^\circ)^E = A^E$.*

Proof. $(A^\circ)^E = \left(\overline{A^c}\right)^\circ = \left(\overline{A^\circ}\right)^c = \left(\overline{A}\right)^c = A^E$. □

Remark 8. The converse of the previous theorem is not true.

Example 3. Let A={3,4}, then $A^E = \left(\overline{\{3,4\}}\right)^c = (\{3,4\})^c = \{1,2\}$.

And $(A^\circ)^E = \left(\overline{\{1,2\}}\right)^\circ = \{1,2,4\}^\circ = \{1,2\}$, thus $(A^\circ)^E = A^E$, but A is not open set.

Theorem 10. *If $(A^\circ)^E = A^E$, then*

1. *A is semiopen set.*
2. *$(A^\circ)^b = A^b$.*

Proof. 1. Since $(A^\circ)^E = A^E$, then $\left(\overline{A^c}\right)^\circ = \left(\overline{A^\circ}\right)^c = \left(\overline{A}\right)^c$. Thus $\left(\overline{A^\circ}\right) = \overline{A}$, but always $A \subseteq \overline{A}$, then $A \subseteq \left(\overline{A^\circ}\right) = \overline{A}$. Hence A is semiopen set by Definition 3, (1).

2. $(A^\circ)^b = \overline{A^\circ} \cap \overline{A^c} = \overline{A} \cap \overline{A^c} = A^b$.

□

Theorem 11. *If A is closed then:*

1. $\left(A^b\right)^\circ \subseteq A$.
2. $\left(A^\circ\right)^b \subseteq A$.
3. $\left(A^E\right)^b \subseteq A$.
4. $\left(A^b\right)^b \subseteq A$.

Proof. Let A be closed, and since $\left(A^b\right)^\circ \subseteq A^b$, then $\left(A^b\right)^\circ \subseteq A$. The same folows for the other parts 2., 3., and 4. □

Remark 9. The converse of part 2., Theorem 11 is not true.

Example 4. Take $A = \{1,4\} \subseteq X = \{1,2,3,4\}$ with topology $\tau = \{\emptyset, X, \{1,2\}, \{3\}, \{1,2,3\}\}$, then $\left(A^\circ\right)^b = \left(\{1,4\}^\circ\right)^b = (\emptyset)^b = \emptyset \subseteq A$, but A is not closed in X.

Theorem 12. *If $\left(A^b\right)^E = \emptyset$, then:*

1. $A^\circ = A^E = \emptyset$.
2. $A^b = \left(A^\circ\right)^E = \left(A^b\right)^\circ = X$.
3. $\left(A^E\right)^\circ = \left(A^\circ\right)^b = \left(A^b\right)^b = \emptyset$.

Proof. 1. If $\left(A^b\right)^E = \emptyset$, then $\left(A^b\right)^c = \emptyset$, hence $A^b = X$. Therefore $A^\circ = A^E = \emptyset$.

2. $\left(A^\circ\right)^E = \left(\overline{A^\circ}\right)^c = \left(\overline{\emptyset}\right)^c = X$.

   With $\left(A^b\right)^\circ = \left(\overline{A}\right)^\circ \cap \left(\overline{A^c}\right)^\circ = (X)^\circ = X$.

3. $\left(A^\circ\right)^b = (\emptyset)^b = \emptyset$, and $\left(A^b\right)^b = (X)^b = \emptyset$.

□

Theorem 13. *If A is clopen subset of X, then:*

1. $(A^b)^b = (A^b)^\circ = (A^E)^b = \emptyset$, *and* $(A^b)^E = X$.

2. $(A^\circ)^E = (A^E)^\circ = A^c$, $(A^E)^E = A$, *and therefore the set* $\{A^{EE}, A^{E\circ}\}$ *forms a partition of* X.

3. *A is regular clopen.*

4. $(A^b)^E$ *is dense in* X.

Proof. 1. Since A is clopen which implies that $A^b = \emptyset$, but $(A^b)^b, (A^b)^\circ$ and $(A^E)^b \subseteq A^b = \emptyset$, the claim follows.

Also, $(A^b)^E = (\emptyset)^c = X$.

2. Since A is clopen we have $(A^E)^\circ = A^E = (\overline{A})^c = A^c$ and hence its clopen, and $(A^\circ)^E = (\overline{A^c})^\circ = (A^c)^\circ = A^c$. Also we have $(A^E)^E = (\overline{A})^\circ = A^\circ = A$. Clearly $\{A^{EE}, A^{E\circ}\}$ forms a partition of X.

3. Since A is clopen, we have $(\overline{A})^\circ = A^\circ = A$, then A is regular open and $(\overline{A^\circ}) = \overline{A} = A$, so A is regular closed. Hence A is regular clopen.

4. $\overline{(A^b)^E} = \overline{(A^b)^c} = \overline{X} = X$. Thus$(A^b)^E$ is a dense in X.

□

Theorem 14. *If* $(A^b)^b = \emptyset$, *then* $(A^b)^\circ = A^b$ *and hence* A^b *is clopen.*

Proof. Since

$$\left(A^b\right)^b = A^b \cap \overline{(A^b)^c} = \emptyset, \text{ then } A^b \subseteq \left(\overline{(A^b)^c}\right)^c = \left(A^b\right)^\circ, \tag{6}$$

but $(A^b)^\circ \subseteq A^b$, so $(A^b)^\circ = A^b$. Also from (6) A^b is an open set, but A^b is also closed, thus A^b is clopen. □

Remark 10. If $A^{bb} = \emptyset$ and $(A^b)^\circ \subseteq A$, then A is closed.

Proof. By the above theorem, $(A^b)^b = \emptyset$, then $(A^b)^\circ = A^b \subseteq A$ which implies that A is closed. □

Lemma 3. *If* $(A^E)^b = \emptyset$. *Then :*

1. $\left(A^E\right)^E = \left(A^E\right)^c$.

2. $\overline{A}$ *is clopen, and hence* A^E*is clopen.*

3. *If* A *is a semiopen set, then* A *is an* α*-open set.*

4. A *is a preopen set.*

5. A^b *is clopen.*

Proof. 1. $\left(A^E\right)^b = \emptyset$, so $\left(\overline{A}\right)^b = \overline{A} \cap \overline{\left(\overline{A}\right)^c} = \emptyset$, which implies that $\overline{A} \subseteq \left(\overline{\left(\overline{A}\right)^c}\right)^c = \left(\overline{A}\right)^\circ$, but $\left(\overline{A}\right)^\circ \subseteq \overline{A}$. Thus $\left(\overline{A}\right)^\circ = \left(A^E\right)^E = \overline{A} = \left(A^E\right)^c$.

2. We have $\left(\overline{A}\right)^\circ = \overline{A}$, then $\overline{A}$ is open, but it is also closed, hence $\overline{A}$ is clopen. Where $A^E = \left(\overline{A}\right)^c$; consequently, A^E is a clopen set.

3. Now, As $\left(\overline{A}\right)^\circ = \overline{A}$, then $\left(\overline{A^\circ}\right)^\circ = \overline{A^\circ}$, since A is semiopen $A \subseteq \overline{A^\circ} = \left(\overline{A^\circ}\right)^\circ$. Thus A is an α-open set.

4. Since $A \subseteq \overline{A} \subseteq \left(\overline{\left(\overline{A}\right)^c}\right)^c = \left(\overline{A}\right)^\circ$, this implies that A is a preopen set.

5. $\left(A^b\right)^\circ = \left(\overline{A}\right)^\circ \cap \left(\overline{A^c}\right)^\circ = \overline{A} \cap \overline{A^c} = A^b$, then A^b is open and it is also closed, hence it is clopen.

□

Theorem 15. *If* A *is semiclosed subset of* X*, and* $\left(A^E\right)^E = \left(A^E\right)^c$*, then* A *is closed.*

Proof. Since $\left(A^E\right)^E = \left(A^E\right)^c$, therefore $\left(\overline{A}\right)^\circ = \overline{A} \subseteq A$. Thus A is closed. □

Theorem 16. *If* $\left(A^\circ\right)^b = \emptyset$*, then*

1. A *is preclosed subset of* X*.*

2. A°*is clopen, and hence* X *has a partition topology.*

Proof. 1. $\left(A^\circ\right)^b = \overline{A^\circ} \cap \overline{A^c} = \emptyset$, then $\overline{A^\circ} \subseteq \left(\overline{A^c}\right)^c = A^\circ \subseteq A$. Hence A is preclosed.

2. By part 1., we have $\overline{A^\circ} \subseteq A^\circ$, but it is always true that $A^\circ \subseteq \overline{A^\circ}$ gives $A^\circ = \overline{A^\circ}$ which implies that $A^\circ$is closed, but it is also open. Hence it is clopen and X has a partition topology.

□

Lemma 4. *If $\left(A^b\right)^\circ = \emptyset$. Then:*

1. *Every preopen is a semiopen subset.*
2. *If A is preopen subset then A is an α- open set.*

Proof. 1. $\left(A^b\right)^\circ = \left(\overline{A}\right)^\circ \cap ((A^\circ)^c)^\circ = \emptyset$, therefore $\left(\overline{A}\right)^\circ \subseteq \left(((A^\circ)^c)^\circ\right)^c = \overline{A^\circ}$.

Since A is preopen, we have $A \subseteq \left(\overline{A}\right)^\circ \subseteq \overline{A^\circ}$, then A is a semiopen subset.

2. By part 1., $\left(\overline{A}\right)^\circ \subseteq \left(\overline{A^\circ}\right)$, so $\left(\overline{A}\right)^\circ \subseteq \left(\overline{A^\circ}\right)^\circ$ but A is preopen subset, then $A \subseteq \left(\overline{A}\right)^\circ \subseteq \left(\overline{A^\circ}\right)^\circ$, thus A is an α-open subset of X.

□

Theorem 17. *If $\left(A^E\right)^E = \left(A^E\right)^c$, and $\left(A^b\right)^\circ = \emptyset$, then:*

1. *A is semiopen.*
2. *Every preclosed is closed.*

Proof. 1. By Lemma 2 we have $\left(A^b\right)^\circ = \emptyset$, then $\left(\overline{A}\right)^\circ \subseteq \overline{(A^\circ)}$, and $\left(A^E\right)^E = \left(A^E\right)^c$, hence $\left(\overline{A}\right)^\circ = \overline{A}$ which implies that $\overline{A} \subseteq \overline{A^\circ}$, but $A \subseteq \overline{A}$, which gives $A \subseteq \overline{A^\circ}$. Hence A is a semiopen subset of X.

2. If A is preclosed, then $\overline{A^\circ} \subseteq A$, therefore $\Rightarrow \overline{A} \subseteq \overline{A^\circ} \subseteq A$. Hence A is closed.

□

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