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ON MIDY'S THEOREM IN BASE b

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Abstract: In 1836 E. Midy proved that if the period of a reciprocal of a prime $p \geq 5$ has even length and is split into two half-periods, then the sum of the halves is a string of 9's. This result has been generalized to any base b. In this note we give a very simple proof of this generalization.

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1. Introduction

According to Dickson [1], E. Midy proved in 1836 that if the period of a reciprocal of a prime $p \geq 5$ has even length and is split into two half-periods then the sum of the halves is a string of 9's. For example, $\frac{1}{7} = 0.\overline{142857}$ with 142+857=999, and

$$\frac{1}{17} = 0.\overline{0588235294117647} \text{ with } 05882352 + 94117647 = 99999999.$$

Midy's theorem holds also for reciprocals of some composite numbers. For example, $\frac{1}{77} = 0.\overline{012987}$ with $012 + 987 = 999$, and

$$\frac{1}{121} = 0.\overline{0082644628099173553719}$$

with

$$00826446280 + 99173553719 = 99999999999.$$

Midy's theorem is true for any base b. For example, in base 8 we have $\frac{1}{19} = [0.\overline{032745}]_8$ with 032+745=777, and in base 6 we have $\frac{1}{13} = [0.\overline{024340531215}]_6$

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with 024340+531215=555555.

2. Midy's Theorem in Base b

Many authors have given proofs of Midy's theorem and its generalization to any base b; see, for example, [2], [3], [4], [5], and [6]. In this paper we give a very simple proof of Midy's theorem in base b based only on modular arithmetic.

We first give a simple method for finding the elements of the period of the reciprocal of a natural number m relatively prime to the base b. This generalizes the lemma in [3].

Lemma 1. *Let m be relatively prime to the base b. Choose $0 \leq c_i < m; i = 1, 2, ..., r$ such that $b^i \equiv c_i$ mod m, where r is the order of b modulo m. Then $\dfrac{1}{m} = [0.\overline{a_1 a_2 ... a_r}]_b$, where $a_i \equiv -m^{-1} c_i$ mod b.*

Proof. For some $0 \leq h_i < m; i = 1, 2, ..., r$, we have the following equations:
$b = h_1 + ma_1$
$bh_1 = h_2 + ma_2$
$bh_2 = h_3 + ma_3$
.
.
.
$bh_{r-1} = h_r + ma_r$.

So $h_1 \equiv b \bmod m$ and for $i = 1, 2, ..., r$, we have $b^{i-1}h_1 = h_i + ma_i + bma_{i-1} + b^2 ma_{i-2} + ... + b^{i-2} ma_2$. Thus modulo m we have $h_i \equiv b^{i-1} h_1 \equiv b^{i-1} b = b^i \equiv c_i$ for $i \geq 2$. Since $0 \leq h_i, c_i < m$, then $h_i = c_i; i = 1, 2, ..., r$. Hence modulo b we have $ma_i \equiv h_i = c_i$ and thus $a_i \equiv -m^{-1} c_i$. □

As an example, consider the period of $\frac{1}{14}$ in base 5. We have $\frac{1}{14} = [0.\overline{013431}]_5$. Then $-m^{-1} \equiv -4 \equiv 1$ mod 5. So $a_i \equiv c_i$ mod 5, where $c_i \equiv 5^i$ mod 14. So $c_1 = 5, c_2 = 11, c_3 = 13, c_4 = 9, c_5 = 3$, and $c_6 = 1$. Thus $a_1 = 0, a_2 = 1, a_3 = 3, a_4 = 4, a_5 = 3$, and $a_6 = 1$. Note that 013+431=444.

Corollary 1. ([3]) *Let m be relatively prime to 10 and choose $0 \leq b_i < m; i = 1, 2, ..., r$ such that $10^i \equiv b_i$ mod m, where r is the order of 10 modulo m. Then $\dfrac{1}{m} = 0.\overline{a_1 a_2 ... a_r}$, where a_i is congruent modulo* 10 *to:* a) $9b_i$ *if $m \equiv 1$ mod* 10*;*

b) $3b_i$ *if* $m \equiv 3$ *mod* 10*;*
c) $7b_i$ *if* $m \equiv 7$ *mod* 10*; and*
d) b_i *if* $m \equiv 9$ *mod* 10*.*

Proof. Modulo 10, if $m \equiv 1$, then $-m^{-1} \equiv -1 \equiv 9$,if $m \equiv 3$, then $-m^{-1} \equiv -7 \equiv 3$, if $m \equiv 7$, then $-m^{-1} \equiv -3 \equiv 7$, and if $m \equiv 9$, then $-m^{-1} \equiv -9 \equiv 1$. □

Now we use our lemma to give a very simple proof of Midy's theorem below in base b.

Theorem 1. *Let b, m, c_i and r as in the above lemma. If $r = 2w$ is even and $b^w \equiv -1$ mod m, then*

$$\frac{1}{m} = [0.\overline{a_1 a_2 ... a_w a_{w+1} a_{w+2} ... a_{2w}}]_b$$

$$\textit{with} \ \ a_i + a_{w+i} = b - 1;\ i = 1, 2, ..., w.$$

Proof. For $i = 1, 2, ..., r$ we have mod m, $c_{w+i} \equiv b^{w+i} = b^w b^i \equiv -b^i \equiv m - c_i$ so $c_{w+i} = m - c_i$. Thus by the above lemma, for $i = 1, 2, ..., w$ we have $a_i + a_{w+i} \equiv -m^{-1} c_i + (-m^{-1}(m - c_i)) \equiv -1 \equiv b - 1$ mod b. Since $0 \leq a_i, a_{w+i} < b$ then $a_i + a_{w+i} = b - 1$. □

Corollary 2. *If p is a prime and the period of $\frac{1}{p}$ in base b has even length, say $r = 2w$, then*
$\frac{1}{p} = [0.\overline{a_1 a_2 ... a_w a_{w+1} a_{w+2} ... a_{2w}}]_b$ *with* $a_i + a_{w+i} = b - 1$*;* $i = 1, 2, ..., w$*.*

Proof. Since $r = 2w$ is the order of b mod p then $b^r = b^{2w} \equiv 1$ mod p. So $(b^w - 1)(b^w + 1) \equiv 0$ mod p. Since $b^w - 1$ is not congruent to 0 mod p, then $b^w + 1 \equiv 0$ mod p. The result now follows immediately from the above theorem. □

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