

REMARK ON THE PALM INTENSITY OF
NEYMAN-SCOTT CLUSTER POINT PROCESSES

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*Dedicated to Professor Yoshihiko Ogata on
his 66th birthday with respect and affection.*

Abstract: The present paper is concerned with notes on the Palm intensity of Neyman-Scott cluster point processes. The Palm intensity is known as the most important and informative second-order characteristic that plays a role of a configurational criterion for point patterns. It is remarkable that its pole at the origin and range of correlation play some intrinsic roles in the point pattern analysis. However, their generic forms cannot be derived due to the inability to explicitly describe the Palm intensity except in typical cluster point processes. This paper provides a sufficient condition for the existence of the pole and a bound on the range of correlation for more general cluster point processes. As applications of our results, we give an exact formula for a variance-area curve of the Neyman-Scott cluster point processes and a remark on a sufficient condition for a Neyman-Scott cluster point process with uniformly bounded diameter to be a connected component Markov point process. The current results play a key role in the identification problem on a superposed Neyman-Scott cluster point process model due to the Palm intensity, which is the most significant pragmatic contribution made in this paper.

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1. Preliminaries

In this paper, the second-order characteristic of Palm conditional intensity of Neyman-Scott cluster point processes is dealt with, which is proportional to a pair-correlation function (or a radial distribution function). As a prerequisite, we assume that readers are familiar with the aforementioned notion (if necessary see [1], [6], [7], [8] and [19] for detailed accounts). Throughout this paper, we restrict attention to the case where an ambient space under consideration is planar. Our results are believed to generalize to arbitrary finite dimensions.

Throughout this paper, an observed window is prescribed to be equipped with a periodic boundary condition, namely, the window is joined on opposite edges to form a torus on which the considered point processes are endowed with uniformity (stationarity) and isotropy.

Let us recall the Palm conditional intensity, which from now on will be abbreviated as *Palm intensity*. The Palm distribution formalizes the concepts of conditioning on a point of the process. Historically, the Palm intensity was developed by C. Palm (1907-1951) for the study of telephone traffic [14] (see also [5, § 8.3.4] for a modern mathematical treatment). The Palm intensity is in fact the intensity function of the Palm probability measure or the Palm probability distribution (so-called the difference process in the sense of [23, p. 24], [24, § 2.3]) denoted by $\Pr^{\mathbf{x}}$ associated with the original point process X at the location \mathbf{x} . In the spirit of [13, p. 464], [22] and [24, § 2.2], the Palm intensity is described as follows.

Definition 1.1.

$$\begin{aligned} \lambda_{\mathbf{o}}(\mathbf{x}) &:= \frac{\Pr^{\mathbf{o}}(\{N(d\mathbf{x}) = 1\})}{\text{Vol}(d\mathbf{x})} \\ &= \frac{\Pr(\{N(d\mathbf{x}) = 1 \mid N(\{\mathbf{o}\}) = 1\})}{\text{Vol}(d\mathbf{x})} \\ &= \frac{\mathbb{E}[N(d\mathbf{x}) \mid N(\{\mathbf{o}\}) = 1]}{\text{Vol}(d\mathbf{x})}, \end{aligned}$$

where N , \mathbf{o} and $d\mathbf{x}$ signify a counting measure, the origin of the Euclidean space and an infinitesimal set containing an arbitrary given point \mathbf{x} , respectively.

To rephrase, the Palm intensity $\lambda_{\mathbf{o}}$ implies the occurrence rate at \mathbf{x} provided that a point is at \mathbf{o} . Let r be the distance from \mathbf{o} to \mathbf{x} . Due to uniformity and isotropy of X , $\lambda_{\mathbf{o}}$ depends only on the distance. Hence we have the polar

coordinate representation with respect to the distance r as

$$\lambda_o(\mathbf{x}) = \lambda_o(r, \theta) = \lambda_o(r), \quad 0 \leq r < \infty, \quad 0 \leq \theta < 2\pi.$$

Definition 1.2. If λ_o of the Neyman-Scott cluster point process satisfies

$$\lambda_o(0) := \lim_{r \downarrow 0} \lambda_o(r) = \infty,$$

then it is said that λ_o admits a pole at \mathbf{o} .

We now denote by r_o the *range of correlation* of λ_o , which is defined as per the following:

Definition 1.3.

$$r_o := \inf_{\lambda_o(r)=\lambda} r,$$

where λ denotes the intensity of an arbitrary given point process.

From now on up to § 4.3, let us define X to be an arbitrary planar Neyman-Scott cluster point process, and we shall concern ourselves with λ_o and r_o of X .

Remark 1.4. r_o implies a geometric quantity of diameter of each closed ball B centred each cluster centre, which contains each cluster of X (so-called *representative cluster* in the spirit of [19, p. 157], which from now on will be denoted by N_0) (see [10, p. 239] for detailed accounts).

The remainder of this section is devoted to reviewing the basic properties of λ_o .

Remark 1.5. It is known that λ_o is of smoothly monotone decreasing form (see [10, p. 220] and the right panel in Fig 1 for its illustration), so that r_o is unique.

Within the scope of X , we denote by μ , ν , and q_τ the intensity of the cluster centres (so-called parent points) distributed according to a uniform Poisson point process, the mean of the i.i.d. Poisson distributed random number of the cluster points (so-called descendants, offspring points or daughter points) isotropically scattered around the respective cluster centres, and the probability

density of the i.i.d. distance emanating from the respective cluster centres to the corresponding cluster points, where τ denotes its parameter set, respectively. Furthermore, q_τ is referred to as the *dispersal kernel*. Throughout the present paper, we assume q_τ to be continuous over $[0, \infty)$ unless otherwise stated.

Theorem 1.6 (e.g., [24]). λ_o is of the following form:

$$\lambda_o(r) = \lambda + \frac{\nu f_\tau(r)}{2\pi r}, \quad 0 \leq r < \infty, \tag{1.1}$$

where λ is given by $\mu\nu$ and f_τ denotes the probability density with respect to the distance r between two arbitrary cluster points in N_0 , which is the derivative of the probability distribution F_τ given by

$$\begin{aligned} \frac{F_\tau(r)}{2} &= \int_0^{\frac{r}{2}} \left\{ \int_{r-r_1}^{r+r_1} h(r, r_1, r_2) q_\tau(r_2) dr_2 \right\} q_\tau(r_1) dr_1 \\ &+ \int_{\frac{r}{2}}^\infty \left\{ \int_{r_1}^{r+r_1} h(r, r_1, r_2) q_\tau(r_2) dr_2 \right\} q_\tau(r_1) dr_1 \tag{1.2} \\ &+ \int_0^{\frac{r}{2}} \left\{ \int_{r_1}^{r-r_1} q_\tau(r_2) dr_2 \right\} q_\tau(r_1) dr_1, \end{aligned}$$

where $r_i, i = 1, 2$ denotes the distance from a cluster centre to the corresponding cluster point and $h(r, r_1, r_2) := \pi^{-1} \arccos((r_1^2 + r_2^2 - r^2)/2r_1r_2)$.

If B admits a uniformly bounded diameter, then X is referred to as a *uniformly bounded cluster*.

2. Statement of Results

In this paper we are concerned with clarifying the two important and interesting quantities of the pole at \mathbf{o} of λ_o and r_o . For given random point patterns and planar curves, provided that the pole exists, an order of the pole implies their fractal dimension. For more details we refer the reader to [13], [16] and [18]. On the other hand, r_o is geometrically characterized by the *limit of the invisible distance of the point pattern interaction in N_0* (see [10, p. 220] and [21, § 3.2.2]). Its invisibility results from overlaps of N_0 's. *Therefore it is worthwhile estimating the quantities $\lambda_o(0)$ and r_o .*

We are now in a position to state our results. The proofs of Propositions 2.1–2.2 and Main Theorem can be found in § 3. The following proposition gives

an estimate of λ_o from below at \mathbf{o} . In [21], we have extended our results to superposed Neyman-Scott cluster point processes.

Proposition 2.1. *Suppose that q_τ is of class $C^r([0, \infty))$, $r \geq 1$. It then holds that*

$$\lambda + \frac{q_\tau(0)^2 \nu}{2\pi} \leq \lambda_o(0) \quad (< \infty) \quad (2.1)$$

whenever λ_o never admits a pole at \mathbf{o} .

The following proposition gives a sufficient condition for the existence of a pole at \mathbf{o} of λ_o .

Proposition 2.2. *Assume that*

$$\lim_{r \downarrow 0} \frac{\int_0^{\frac{r}{2}} \left\{ \int_{r_1}^{r-r_1} q_\tau(r_2) dr_2 \right\} q_\tau(r_1) dr_1}{r^2} = \infty.$$

Then λ_o admits a pole at \mathbf{o} .

Example 2.3. Let q_τ be an exponential distribution with parameter β , namely, $q_\beta(r) = \beta \exp(-\beta r)$, $\beta > 0$, $0 \leq r < \infty$. From now on we will be referring to this model as *exponential type model*, a realization of which is illustrated in the right panel of Fig.1. Despite λ_o of the exponential type model not being specified, by virtue of Proposition 2.2 we see that the exponential type model admits a pole at \mathbf{o} .

The following theorem gives a bound on r_o .

Main Theorem. *Suppose that q_τ is bounded from above. It then holds that*

$$\frac{1}{2C} \leq r_o,$$

where $C := \sup_{r \geq 0} q_\tau(r)$. Suppose furthermore that an arbitrary given q_τ is monotonic. Then r_o is further estimated as per the following:

$$\begin{cases} r_o \leq \sqrt{2}/q_\tau(0), \\ q_\tau(0) > 0, \text{supp } q_\tau \in C_0^r([0, \infty)), & q_\tau: \text{non-decreasing;} \\ r_o \sqrt{q_\tau(r_o/2)q_\tau(r_o)} \leq \sqrt{2}, & q_\tau: \text{non-increasing.} \end{cases} \quad (2.2)$$

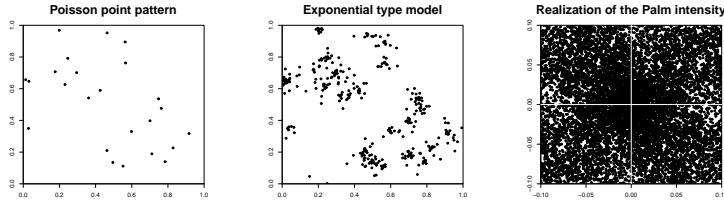


Figure 1: The left, centre and right panels exhibit the simulations of a Poisson point process with $\mu = 30.0$, the corresponding exponential type model with $(\mu, \nu, \beta) = (30.0, 15.0, 30.0)$ and its Palm intensity, respectively. We know from the posterior half of the Example 2.3 that $\lambda_o(0) = \infty$, namely, λ_o is densest at \mathbf{o} (see the right panel in Fig 1). The above graphics operation is implemented by [23].

Corollary 2.4. *Suppose that q_τ satisfies the monotonic assumption given by (2.2). Then X becomes a uniformly bounded cluster.*

Remark 2.5. Note that it is impossible for one to find a diameter of B , because each N_0 overlaps and therefore, by Neyman-Scott cluster point process definition, neither cluster centre lies in X . Nevertheless one can see by virtue of Corollary 2.4 whether X is a uniformly bounded cluster or not.

Example 2.6. We apply our Main Theorem to the exponential type model with $\beta = 30.0$. Although one cannot find r_o from the right panel of Fig. 1, by virtue of our Main Theorem we get $1/60 \approx 0.0167 \leq r_o$.

Incidentally, [21] has exemplified r_o of a (modified) Thomas process and a Matérn cluster process, which are typical Neyman-Scott cluster processes to which we refer the reader for further accounts.

Although the existence of the pole and estimation of the range of correlation have not been found anywhere in the literature, our results are rather easily derived. The main purpose of this paper is to fill this gap.

To this end by virtue of our results, *one has only to consider an arbitrary given q_τ . In other words, one need not derive the Palm intensity directly.*

3. Proofs of the Results

As a rule, throughout the proofs straightforward calculation steps are omitted.

Proof of Proposition 2.1. We see by L'Hospital's theorem that

$$\lim_{r \downarrow 0} \lambda_o(r) = \lambda + \frac{\nu}{\pi} \lim_{r \downarrow 0} \frac{F_\tau(r)}{r^2}$$

whenever $\lambda_o(0) < \infty$.

Let us estimate from below the left-hand side of (1.2). We readily obtain

$$\frac{F_\tau(r)}{2} \geq \int_0^{\frac{r}{2}} \left\{ \int_{r_1}^{r-r_1} q_\tau(r_2) dr_2 \right\} q_\tau(r_1) dr_1, \quad 0 \leq r < \infty. \quad (3.1)$$

Now it follows from the hypothesis concerning q_τ that

$$\begin{aligned} \infty &> \frac{q_\tau(0)^2}{4} \\ &= \lim_{r \downarrow 0} \frac{\int_0^{\frac{r}{2}} \frac{\partial}{\partial r} q_\tau(r-r_1) q_\tau(r_1) dr_1 + q_\tau(r/2)^2 / 2}{2} \\ &= \lim_{r \downarrow 0} \frac{\int_0^{\frac{r}{2}} q_\tau(r-r_1) q_\tau(r_1) dr_1}{2r} \quad \text{by L'Hospital's theorem.} \end{aligned} \quad (3.2)$$

Since (3.2) converges, by L'Hospital's theorem we have

$$\lim_{r \downarrow 0} \frac{\int_0^{\frac{r}{2}} \left\{ \int_{r_1}^{r-r_1} q_\tau(r_2) dr_2 \right\} q_\tau(r_1) dr_1}{r^2} = (3.2),$$

from which (2.1) holds. The proof is finished. \square

Proof of Proposition 2.2. Taking the contraposition of the proof of Proposition 2.1 yields the desired result. \square

Proof of Main Theorem. First of all, by Definition 1.3 and (1.1), and also from Remark 1.5 we notice that f_τ is the probability density with support r_o , i.e.,

$$F_\tau(r_o) = \int_0^{r_o} f_\tau(r) dr = 1. \quad (3.3)$$

By (1.2), the left-hand side of (3.3) is bounded from above as per the following:

$$\begin{aligned} \frac{F_\tau(r_o)}{2} &\leq \int_0^{\frac{r_o}{2}} \left\{ \int_{r_o-r_1}^{r_o+r_1} q_\tau(r_2) dr_2 \right\} q_\tau(r_1) dr_1 \\ &\quad + \int_{\frac{r_o}{2}}^\infty \left\{ \int_{r_1}^{r_o+r_1} q_\tau(r_2) dr_2 \right\} q_\tau(r_1) dr_1 \\ &\quad + \int_0^{\frac{r_o}{2}} \left\{ \int_{r_1}^{r_o-r_1} q_\tau(r_2) dr_2 \right\} q_\tau(r_1) dr_1 \\ &= \int_0^\infty \left\{ \int_{r_1}^{r_o+r_1} q_\tau(r_2) dr_2 \right\} q_\tau(r_1) dr_1 \\ &\leq Cr_o. \end{aligned}$$

By substituting the result back to (3.3) we get the proof for the lower bound on r_o .

Next let us estimate from below $F_\tau(r_o)$ in a fashion similar to (3.1). In particular, in the case when q_τ is non-decreasing, $\text{supp } q_\tau$ must be compact because q_τ is a probability density and we immediately have

$$\frac{F_\tau(r_o)}{2} \geq \frac{q_\tau(0)^2 r_o^2}{4},$$

whence plugging into (3.3) the assertion follows. On the other hand, in the case when q_τ is non-increasing, in like manner

$$\frac{F_\tau(r_o)}{2} \geq \frac{r_o^2 q_\tau(r_o/2) q_\tau(r_o)}{4}$$

holds. This completes the proof of our Main Theorem. □

Proof of Corollary 2.4. We see at once that (2.2) may give us an upper bound on r_o as desired. □

4. Some Applications

In this section, we shall apply the current results to a variance-area curve of X as well as to the assumption that a uniformly bounded cluster becomes a connected component Markov point process. The identification problem is well-known within different fields, for example [12], which has a well-known

application to differential geometry, stochastic geometry as well as point processes, in which [11] has conjectured that, in general, the human eye cannot easily discriminate between point patterns that have the same second-order characteristics. Subsequent works by [2] have given counterexamples to Julesz's conjecture. The most significant application is to give a remark on a sufficient condition for the identification problem on a superposed Neyman-Scott cluster point process model due to λ_o of the model. By applying pseudo-likelihood analysis that makes use of the Palm intensity [22], recently U. Tanaka and Y. Ogata have solved the identification problem of the superposed Neyman-Scott cluster point process model.

4.1. Application to Variance-Area Curve

As a preceding study on the estimate of the variance of X (sometimes called the *variance-area curve*) comparing two Neyman-Scott cluster point processes, [15] has discussed its relative estimate. The purpose of this subsection is to give an exact formula for the variance-area curve.

Under a Radon-Nikodym theorem assumption for a second-factorial moment measure (see [17, (8) in p. 7] for detailed accounts), the variance-area curve is given by

$$\text{Var } N(B) = \lambda \left(\text{Vol}(B) + \nu \int_B \int_B f_\tau(x-y) dx dy \right),$$

where x and y are arbitrary cluster points in N_0 (see [9, p. 65]). In view of Remark 1.4 and by virtue of (3.3), the variance-area curve is given by

$$\text{Var } N(B) = \lambda \left(\frac{\pi r_o^2}{4} + 2\pi\nu \right).$$

4.2. Application to Connected Component Markov Point Process

The purpose of this subsection is to give a remark on the assumption on the uniformly bounded cluster, which for a uniformly bounded cluster is assumed to be a connected component Markov point process (see [4, Sect. 1] and the references therein for detailed accounts).

Since an arbitrary Neyman-Scott cluster point process is a Poisson cluster point process, from [4, Theorem 2.1] ([3, Theorem 1]) we see at once the following theorem:

Theorem 4.1. *Suppose that X has q_τ satisfying measurability and positivity conditions, which are suppressed here, and $N_0 \neq \emptyset$ a.s. and*

$$N_0 \subset B_R(a_i) \quad \text{a.s.} \quad \exists R < \infty \quad \forall i, \quad (4.1)$$

where each $B_R(a_i)$ signifies a closed ball centred each cluster centre a_i of radius R . Then X becomes a connected component Markov point process at distance $2R$.

Our remark on Theorem 4.1 is stated as per the following:

Remark 4.2. (4.1) has been assumed in [3, Theorem 1] and [4, Theorem 2.1]. Nevertheless, from the pragmatic point of view, we see by Remark 2.5 that investigating and/or satisfying (4.1) is quite difficult. For all practical purposes, our assumption (2.2) facilitates checking whether or not (4.1) holds. Indeed, (4.1) still holds true by Corollary 2.4 if the assumption is satisfied.

4.3. Application to Identification Problem

Throughout this subsection we denote by X a superposed Neyman-Scott cluster point process. We would like to conclude this paper with an argument on the identification problem on X due to λ_o of X . Propositions 2.1–2.2 and the Main Theorem play a significant role in ascertaining whether the identification problem happens or not. In fact, as described in § 5, one cannot derive an explicit form of $\lambda_o(r)$, although our results enable one to inquire closely into its local behaviour for the pole and the range of correlation. Thus, in order to discriminate two superposed Neyman-Scott cluster point processes, by virtue of the current results one has only to compare the two quantities: $\lambda_o(0)$ and r_o instead of explicit forms of $\lambda_o(r)$ (see also [21, Remark 6.2]).

5. Concluding Remarks and Discussion

We would like to point out that the analytically closed form of F_τ is not available because of its complicated form as given by (1.2), and so is its derivative f_τ (see [24, p. 47]). Thus *to get our results we have no choice but to employ the bounds on F_τ .*

Lastly, as mentioned in Sect. 2, [16] and [18], since it is interesting to estimate an order of the pole that implies the fractal dimension, as a companion paper to the present one, we will tackle this problem elsewhere.

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