International Journal of Applied Mathematics

Volume 26 No. 4 2013, 511-514

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

doi: http://dx.doi.org/10.12732/ijam.v26i4.9

ON ASYMPTOTIC ESTIMATIONS OF THE q-POCHHAMMER SYMBOLS AT $q = 1$

István Mező

Departamento de Matemática
Escuela Politécnica Nacional
Ladrón de Guevara E11-253, Quito, ECUADOR

Abstract: Watson provided an asymptotic estimation of the Euler function $(q;q)_\infty$ when $q \to 1-$. In this short note we reprove and extend his results using Gosper's q-trigonometric functions.

AMS Subject Classification: 11F03, 33F05

Key Words: q-Pochhammer symbol, Gosper's q-trigonometric functions, Euler modular function

1. The Estimations of Watson

The Euler function ϕ is defined as the infinite product

$$\phi(q) = (1-q)(1-q^2)(1-q^3)\cdots \quad (|q|<1).$$

More generally, the q-Pochhammer symbol $(a;q)_\infty$ is an infinite product

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots \quad (|q|<1).$$

Hence $\phi(q) = (q;q)_\infty$. It is obvious that $\phi(1-) = 0$, but the speed of the convergence is harder to see. G. N. Watson proved the next estimation [3]:

$$\phi(q) = (q;q)_\infty = \frac{\sqrt{2\pi}}{\sqrt{1-q}} \exp\left(\frac{\zeta(2)}{\log q}\right)(1+o(1)).$$

Received: September 16, 2013

Using the straightforward relation $(q;q^2)_\infty = \frac{(q;q)_\infty}{(q^2;q^2)_\infty}$, we immediately have an estimation for $(q;q^2)_\infty$ as well:

$$(q;q^2)_\infty = \sqrt{2}\exp\left(\frac{1}{2}\frac{\zeta(2)}{\log q}\right)(1+o(1)). \tag{1}$$

We reprove the Watson estimation using the so-called q-Raabe formula, and then we extend these results.

2. A Shorthand Proof of Watson's Estimation

We use a formula recently published by the present author in [5]:

$$\int_0^1 \log\Gamma_q(x)dx = \frac{\zeta(2)}{\log q} + \log\sqrt{\frac{q-1}{\sqrt[6]{q}}} + \log(q^{-1};q^{-1})_\infty.$$

Here Γ_q is the Jackson q-Gamma function [2]. This relation stands for $q > 1$, but on the right the q-Pochhammer symbol has parameter $q < 1$. As $q \to 1+$, the left hand side tends to $\log(\sqrt{2\pi})$, thanks to the integral formula

$$\int_0^1 \log\Gamma(x)dx = \log\sqrt{2\pi}$$

of Raabe [1]. So the left hand side of the penultimate formula is asymptotically $\log(\sqrt{2\pi})(1+o(1))$. Then a simple rearrangement and variable change $q \to 1/q$ gives the result of Watson. (However, we note that Watson described this estimation in a different but equivalent form.)

3. New q-Pochhammer Symbol Estimations

We shall prove the next estimations as $q \to 1-$:

$$(q;q^{2^n})_\infty(q^{2^n-1};q^{2^n})_\infty = 2\sin\left(\frac{\pi}{2^n}\right)\exp\left(\frac{1}{2^{n-1}}\frac{\zeta(2)}{\log q}\right)(1+o(1)) \; (n>0), \tag{2}$$

$$(q^{2^{n-1}-1};q^{2^n})_\infty(q^{2^{n-1}+1};q^{2^n})_\infty = 2\cos\left(\frac{\pi}{2^n}\right)\exp\left(\frac{1}{2^{n-1}}\frac{\zeta(2)}{\log q}\right)(1+o(1)) \; (n>1). \tag{3}$$

The next special cases are easy to calculate

$$
\begin{aligned}
(q;q^4)_\infty(q^3;q^4)_\infty &= \sqrt{2}\exp\left(\frac{1}{2}\frac{\zeta(2)}{\log q}\right)(1+o(1)), \qquad (4)\\
(q;q^8)_\infty(q^7;q^8)_\infty &= \sqrt{2-\sqrt{2}}\exp\left(\frac{1}{4}\frac{\zeta(2)}{\log q}\right)(1+o(1)),\\
(q^3;q^8)_\infty(q^5;q^8)_\infty &= \sqrt{2+\sqrt{2}}\exp\left(\frac{1}{4}\frac{\zeta(2)}{\log q}\right)(1+o(1)).
\end{aligned}
$$

To prove these estimations, we recall the definition of the Gosper q-trigonometric functions [4, 6]:

$$
\sin_q(\pi z) = q^{(z-1/2)^2}\frac{(q^{2z};q^2)_\infty(q^{2-2z};q^2)_\infty}{(q;q^2)^2_\infty} \quad (0<q<1), \qquad (5)
$$

$$
\cos_q(\pi z) = q^{z^2}\frac{(q^{1-2z};q^2)_\infty(q^{2z+1};q^2)_\infty}{(q;q^2)^2_\infty} \quad (0<q<1). \qquad (6)
$$

These functions tend to the classical ones as $q\to 1-$.

Now we prove (2). Substituting $z=\frac{1}{2^n}$ into (5) and getting closer to 1 with q, we approximately have that

$$
(1+o(1))\sin\left(\frac{\pi}{2^n}\right) = \frac{(q^{1/2^{n-1}};q^2)_\infty(q^{2-1/2^{n-1}};q^2)_\infty}{(q;q^2)^2_\infty}.
$$

Taking the substitution $q\to q^{2^{n-1}}$:

$$
(1+o(1))\sin\left(\frac{\pi}{2^n}\right) = \frac{(q;q^{2^n})_\infty(q^{2^n-1};q^{2^n})_\infty}{(q^{2^{n-1}};q^{2^n})^2_\infty}.
$$

Finally, we can estimate the denominator by (1) to finalize the proof of (2). The proof of (3) is the same, but we need to use the q-cosine function (6).

We note that by using the substitution $z=\frac{1}{3}$ in (5) and (6) respectively, we can easily deduce the next estimations, too:

$$
\begin{aligned}
(q;q^6)_\infty(q^5;q^6)_\infty &= \exp\left(\frac{1}{3}\frac{\zeta(2)}{\log q}\right)(1+o(1)),\\
(q^2;q^6)_\infty(q^4;q^6)_\infty &= \sqrt{3}\exp\left(\frac{1}{3}\frac{\zeta(2)}{\log q}\right)(1+o(1)).
\end{aligned}
$$

There is one more thing what we can do. Estimation (4) can be separated by using the definition of the q-gamma function:

$$
\Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z} \quad (0<q<1).
$$

Then using q^4 in place of q and substituting $z = 1/4$ and $z = 3/4$, we easily get the next estimations:

$$(q;q^4)_\infty = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{4}\right)}(1-q^4)^{1/4}\exp\left(\frac{1}{4}\frac{\zeta(2)}{\log q}\right)(1+o(1)).$$

$$(q^3;q^4)_\infty = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{3}{4}\right)}(1-q^4)^{-1/4}\exp\left(\frac{1}{4}\frac{\zeta(2)}{\log q}\right)(1+o(1)).$$

References

[1] T. Amdeberhan, M.W. Coffey, O. Espinosa, C. Koutschan, D.V. Manna, V.H. Moll, Integrals of powers of loggamma, *Proc. Amer. Math. Soc.*, **139**, No. 2 (2010), 535-545.

[2] G.E. Andrews, R. Askey, R. Roy, *Special Functions*, Cambridge Univ. Press (2000).

[3] B. Gordon, R.J. McIntosh, Some eighth order Mock Theta functions, *J. London Math. Soc.*, **62** (2000), 321-335.

[4] R.W. Gosper, Experiments and discoveries in q-trigonometry, In: *Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics* (Ed-s: F.G. Garvan and M.E.H. Ismail), Kluwer, Dordrecht, Netherlands (2001), 79-105.

[5] I. Mező, A q-Raabe formula and an integral of the fourth Jacobi theta function, *J. Number Th.*, **133**, No. 2 (2012), 692-702.

[6] I. Mező, Duplication formulae involving Jacobi theta functions and Gosper's q-trigonometric functions, *Proc. Amer. Math. Soc.*, **141**, No. 7 (2013), 2401-2410.