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JORDAN-FOX DERIVATION ON GROUP RINGS

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Abstract: In this paper we introduce Jordan-Fox derivation on a group ring RG and give some sufficient conditions under which the Jordan-Fox derivation is Fox derivation.

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1. Introduction

Let RG be the group ring of a group G over a ring with unitary R. We shall denote by $Z(R)$ the center of a ring R. We have a ring homomorphism (the augmentation map) defined by $R^\varepsilon = R$ and $G^\varepsilon = 1$. An additive mapping $' : RG \longrightarrow RG$ will be called a Fox derivation, if $(ab)' = a'b^\varepsilon + ab'$ holds for all pairs $a, b \in RG$, see [1]. We call an additive mapping $' : RG \longrightarrow RG$ a Jordan-Fox derivation, if $(a^2)' = a'a^\varepsilon + aa'$ holds for all $a \in RG$. Obviously, every Fox derivation is Jordan-Fox derivation. The converse is in general not true. Our main result in this paper is the following theorem.

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Theorem 1.1. *Let R be a prime ring with characteristic different from two, G be a group that has no finite normal subgroup $\neq \{1\}$, RG be a group ring of G over R, ε be augmentation map, and $' : RG \longrightarrow RG$ be a Jordan-Fox derivation. If $a^\varepsilon, b^\varepsilon \in Z(R)$ implies $b'a^\varepsilon + ba' - a'b^\varepsilon - ab' = 0$, then $'$ is a Fox derivation.*

2. Main Results

In preparation for the proof of Theorem 1.1, we start with the following propositions.

Proposition 2.1. *Let RG be a group ring such that $charR \neq 2$. If $'$ is a Jordan-Fox derivation of RG, then for all $a, b \in RG$ we have:*

(i) $(ab + ba)' = a'b^\varepsilon + ab' + b'a^\varepsilon + ba.'$

(ii) $(aba)' = a'b^\varepsilon a^\varepsilon + ab'a^\varepsilon + aba'$.

(iii) $(abc + cba)' = a'b^\varepsilon c^\varepsilon + ab'c^\varepsilon + abc' + c'b^\varepsilon a^\varepsilon + cb'a^\varepsilon + cba'$.

Proof. Since $(a^2)' = a'a^\varepsilon + aa'$, by replacing a by $a + b$ we obtain:

$$\begin{aligned} ((a+b)^2)' &= (a+b)'(a+b)^\varepsilon + (a+b)(a+b)' \\ &= (a'+b')(a^\varepsilon + b^\varepsilon) + (a+b)(a'+b') \\ &= a'a^\varepsilon + a'b^\varepsilon + b'a^\varepsilon + b'b^\varepsilon + aa' + ab' + ba' + bb'. \end{aligned} \tag{1}$$

On the other hand,

$$\begin{aligned} ((a+b)^2)' &= (a^2 + ab + ba + b^2)' \\ &= (a^2)' + (ab)' + (ba)' + (b^2)' \\ &= a'a^\varepsilon + aa' + (ab)' + (ba)' + b'b^\varepsilon + bb'. \end{aligned} \tag{2}$$

By comparing (1) and (2), (i) holds. Now let $w = (a(ab + ba) + (ab + ba)a)$ therefore by (i) we obtain

$$\begin{aligned} w' &= a'(ab+ba)^\varepsilon + a(ab+ba)' + (ab+ba)'a^\varepsilon + (ab+ba)a' \\ &= a'a^\varepsilon b^\varepsilon + a'b^\varepsilon a^\varepsilon + aa'b^\varepsilon + a^2b' + ab'a^\varepsilon + aba' \\ &\quad + a'b^\varepsilon a^\varepsilon + ab'a^\varepsilon + b'(a^\varepsilon)^2 + ba'a^\varepsilon + aba' + baa'. \end{aligned} \tag{3}$$

On the other hand,

$$\begin{aligned} w' &= ((a^2b+ba^2)+2aba)' = (a^2b)' + (ba^2)' + 2(aba)' \\ &= (a^2)'b^\varepsilon + a^2b' + b'(a^2)^\varepsilon + b(a^2)' + 2(aba)' \qquad (4) \\ &= (a'a^\varepsilon + aa')b^\varepsilon + a^2b' + b'(a^\varepsilon)^2 + b(a'a^\varepsilon + aa') + 2(aba)' \\ &= a'a^\varepsilon b^\varepsilon + aa'b^\varepsilon + a^2b' + b'(a^\varepsilon)^2 + ba'a^\varepsilon + baa' + 2(aba)'. \end{aligned}$$

By comparing (3) and (4), (iii) holds, since RG is of characteristic not 2. At last, by replacing a by $a+c$ in (ii), we have

$$\begin{aligned} ((a+c)b(a+c))' &= a'b^\varepsilon a^\varepsilon + ab'a^\varepsilon + aba' + c'b^\varepsilon c^\varepsilon + cb'c^\varepsilon + cbc' \\ &+(abc+cba)'. \qquad (5) \end{aligned}$$

On the other hand,

$$\begin{aligned} ((a+c)b(a+c))' &= (a+c)'b^\varepsilon(a+c)^\varepsilon + (a+c)b'(a+c)^\varepsilon \\ &+(a+c)b(a+c)' \\ &= a'b^\varepsilon a^\varepsilon + a'b^\varepsilon c^\varepsilon + c'b^\varepsilon a^\varepsilon + c'b^\varepsilon c^\varepsilon \qquad (6) \\ &+ab'a^\varepsilon + ab'c^\varepsilon + cb'c^\varepsilon + cb'a^\varepsilon \\ &+aba' + abc' + cba' + cbc'. \end{aligned}$$

By comparing (5), (6) we deduce (iii). □

For any Jordan fox derivation $'$ we shall write a^b for

$$(ab)' - a'b^\varepsilon - ab'.$$

Corollary 2.2. $a^b = -b^a$.

Corollary 2.3. $a^{b+c} = a^b + a^c$.

Proof.

$$\begin{aligned} a^{b+c} &= (a(b+c))' - a'(b+c)^\varepsilon - a(b+c)' \\ &= (ab+ac)' - a'b^\varepsilon - a'c^\varepsilon - ab' - ac' \\ &= (ab)' - a'b^\varepsilon - ab' + (ac)' - a'c^\varepsilon - ac' \\ &= a^b + a^c. \end{aligned}$$

Proposition 2.4. *Let RG be a group ring, R be a ring of characteristic not 2 and $'$: $RG \longrightarrow RG$ be a Jordan-Fox derivation. In this case for all $a, b, r \in RG$ we have:*

$$a^b r^\varepsilon(a^\varepsilon b^\varepsilon - b^\varepsilon a^\varepsilon) + (ab-ba)ra^b = 0.$$

Proof. Set $w = abrba + barab$. Then by (ii) of Proposition 2.1, we have:

$$\begin{aligned} w' &= (a(brb)a + b(ara)b)' \\ &= a'(brb)^{\varepsilon}a^{\varepsilon} + a(brb)'a^{\varepsilon} + a(brb)a' \\ &+b'(ara)^{\varepsilon}b^{\varepsilon} + b(ara)'b^{\varepsilon} + b(ara)b' \\ &= a'(brb)^{\varepsilon}a^{\varepsilon} + ab'r^{\varepsilon}b^{\varepsilon}a^{\varepsilon} + abr'b^{\varepsilon}a^{\varepsilon} \\ &+abrb'a^{\varepsilon} + a(brb)a' + b'(ara)^{\varepsilon}b^{\varepsilon} \\ &+ba'r^{\varepsilon}a^{\varepsilon}b^{\varepsilon} + bar'a^{\varepsilon}b^{\varepsilon} + bara'b^{\varepsilon} + b(ara)b'. \end{aligned} \tag{7}$$

On the other hand by (iii) of Proposition 2.1, we obtain:

$$\begin{aligned} w' &= ((ab)r(ba) + (ba)r(ab))' \\ &= (ab)'r^{\varepsilon}(ba)^{\varepsilon} + abr'(ba)^{\varepsilon} + abr(ba)' + (ba)'r^{\varepsilon}(ab)^{\varepsilon} \\ &+(ba)r'(ab)^{\varepsilon} + (ba)r(ab)'. \end{aligned} \tag{8}$$

By comparing (7) and (8) the proof is complete. □

Proposition 2.5. *Let G be a group ring, R be a prime ring of characteristic not 2 and G be an abelian group that has no finite normal subgroup $\neq \{1\}$. Let $a, b \in RG$ and $ar^{\varepsilon}b^{\varepsilon} + bra = 0$ for all $r \in RG$. Then $a = 0$ or $b^{\varepsilon} = 0$.*

Proof. Replacing r by sbt with $s, t \in RG$ we get $as^{\varepsilon}b^{\varepsilon}t^{\varepsilon}b^{\varepsilon} + bsbta = 0$. But $as^{\varepsilon}b^{\varepsilon} = -bsa$ and $bta = -at^{\varepsilon}b^{\varepsilon}$; substituting these we get $-bsat^{\varepsilon}b^{\varepsilon} - bsat^{\varepsilon}b^{\varepsilon} = 0$, that is, $2b(RG)a(RG)^{\varepsilon}b^{\varepsilon} = 0$. Since char $RG \neq 2$ and RG is prime by [2], this immediately gives $a = 0$ or $b^{\varepsilon} = 0$. □

Proof of Theorem 1.1. Let a and b be fixed elements from RG. If $(ab-ba)^{\varepsilon} \neq 0$, then from Propositions 2.4 and 2.5, we obtain that $a^b = 0$. If $(ab - ba)^{\varepsilon} = 0$ and $(ac-ca)^{\varepsilon} \neq 0$ for some $c \in R$, but $(br-rb)^{\varepsilon} = 0$ for any $r \in R$. Then $a^c = 0$ and $a^{b+c} = 0$ thus $a^b = 0$. It remains to prove $(br - rb)^{\varepsilon} = 0$ and $(ar - ra)^{\varepsilon} = 0$ for any $r \in R$. In this case $a^{\varepsilon}, b^{\varepsilon} \in Z(k)$ so $b'a^{\varepsilon} + ba' - a'b^{\varepsilon} - ab' = 0$. Therefore we deduce the result by (i) in Proposition 2.1. □

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