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ON THE COMPLETENESS OF EXCEPTIONAL ORTHOGONAL POLYNOMIALS IN QUANTUM SYSTEMS

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Abstract: Several authors [1, 2, 3, 4, 5, 6] have found some types of non classical orthogonal polynomials, called as Exceptional orthogonal polynomials (EOP), while solving some quantum mechanical systems. Especially, it is Quesne [8, 9] who first observed the presence of a relationship between the exceptional orthogonal polynomials, the Darboux transformation and the shape invariant potentials in quantum mechanics. In this article we demonstrate explicitly the completeness property (in weighted $\mathcal{L}^2$ space) of Exceptional orthogonal polynomials (EOP) [2] in connection with the quantum mechanical states of some categories of well-known quantum mechanical systems.

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1. Introduction

In recent years there has been a growing interest in the construction of new exactly solvable potentials within the framework of supersymmetric quantum mechanics (SUSYQM)/Darboux transformation [10, 11, 12]. This has resulted in generating new families of isospectral potentials. In many cases, solutions of such the potentials are given in terms of non classical orthogonal polynomials.

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Such polynomials are generally called as EOP. Very recently certain families of EOPs have been proposed [1, 3, 5, 6, 8, 14, 15]. With the discovery of such new EOP's, it has gained great attention to the proof of completeness of those polynomials sequences related to some quantum mechanical systems in the quantum mechanical Hilbert space with well defined measure. This has been a challenging issue nowadays. The conventional technique, the most powerful Isometric transform method is also not at all useful to prove the completeness of the new EOP families in the regime of some well known quantum mechanical systems. In this article our objective is to explicitly demonstrate the completeness of a certain type of EOP obtained via Supersymmetric quantum mechanics in Ref [2]. To the best of our knowledge, this elegant approach to deal with the completeness of such exceptional orthogonal families, very important from the point of view of past and planned applications, has not been studied yet.

2. Completeness of the EOP Families Associated to the Quantum System with Unbroken Supersymmetry

We have found in [2] the new set of polynomials family called exceptional orthogonal polynomials (EOP's) and are denoted by

$$q_{n+1}(x) = u(x)H_{n+1}(x) + u'(x)H_n(x) = (1+2x^2)H_{n+1}(x) + 4xH_n(x), q_0 = 1.$$

We are to show that the polynomial sequence $\{q_n\}_{n=0}^{\infty}$ is complete in weighted Hilbert space $\mathcal{L}^2[\Re, w(x)]$. In order to prove the completeness of those EOP's we begin with generating function of $q_{n+1}(x)$ is

$$\psi(x,t) = \sum_{n\geq 0}^{\infty} \frac{q_{n+1}(x)}{n!}\, t^n = e^{2xt-t^2}[4x^3 + 6x - 2t(1+2x^2)], \tag{1}$$

(using the relation $e^{2zx-z^2} = \sum_{n\geq 0} \frac{H_n(x)z^n}{n!}$). We rephrase the generating function

$$\psi(x,t) = e^{2xt-t^2}[4x + 2(x-t)(1+2x^2)]. \tag{2}$$

The weight function is $w(x) = \frac{e^{-x^2}}{(1+2x^2)^2}$. Let $f \in C_c^{\infty}(\Re)$, space of infinitely differentiable functions with compact support on $\Re$ and suppose that

$$< f(x), q_n(x) >= \int_{\Re} f(x)q_n(x)w(x)dx = 0 \;\forall\; n \geq 0$$

$$\Rightarrow \int_{\Re} f(x)\psi(x,t)w(x)dx = 0, \tag{3}$$

as $\psi(x,t)$ generates all polynomial functions. It is worthy to note that this EOP families can be shown to be complete iff $f(x) = 0$. Now,

$$\int_{\Re} f(x)\psi(x,t)w(x)dx$$

$$= \int_{\Re} f(x)\frac{e^{-x^2}}{(1+2x^2)^2}e^{2xt-t^2}[4x + 2(x-t)(1+2x^2)]dx$$

$$= \int_{\Re} \frac{f(x)}{(1+2x^2)^2}e^{-(x-t)^2}[4x + 2(x-t)(1+2x^2)]dx$$

$$= \int_{\Re} \frac{4xf(x)e^{-(x-t)^2}}{(1+2x^2)^2}dx + \int_{\Re} \frac{2f(x)(x-t)e^{-(x-t)^2}}{1+2x^2}dx$$

$$= \int_{\Re} \frac{4xf(x)e^{-(x-t)^2}}{(1+2x^2)^2}dx - \int_{\Re} \frac{f(x)}{(1+2x^2)}\frac{d}{dx}[e^{-(x-t)^2}]dx$$

$$= \int_{\Re} \frac{4xf(x)e^{-(x-t)^2}}{(1+2x^2)^2}dx + \int_{\Re} \frac{d}{dx}[\frac{f(x)}{1+2x^2}]e^{-(x-t)^2}$$

[using the important property in $C_c^\infty(\Re)$ that, for $f, g \in C_c^\infty(\Re)$, $\int_{\Re} f(x)g'(x)dx = -\int_{\Re} f'(x)g(x)dx$]

$$= \int_{\Re} \frac{4xf(x)e^{-(x-t)^2}}{(1+2x^2)^2}dx + \int_{\Re} \frac{(1+2x^2)f'(x) - 4xf(x)}{(1+2x^2)^2}e^{-(x-t)^2}$$

$$= \int_{\Re} \frac{f'(x)}{(1+2x^2)}e^{-(x-t)^2}dx.$$

Hence by hypothesis,

$$\int_{\Re} \frac{f'(x)}{(1+2x^2)}e^{-(x-t)^2}dx = 0 \qquad \forall\, t \in \Re, \tag{4}$$

i.e. $(f_1 * f_2)(t) = 0$, where $f_1(x) = \dfrac{f'(x)}{1+2x^2}$ and $f_2(x) = e^{-x^2}$. Now we refer to the following well known theorem.

Theorem. *Let $f \in C_c^\infty(\Re)$, the space of infinitely differentiable function with compact support on $\Re$, and Gaussian one is $\wedge(x) = e^{-x^2}$. If $f * \wedge(x) = 0$,* *then* $f = 0$. *Here* $'*'$ *stands for the convolution operator.* [*It is because of the fact that when we apply Fourier transform on both sides, then we have* $\widehat{f}\widehat{\wedge(p)} = 0 \Rightarrow \widehat{f} = 0 \Rightarrow f = 0$.]

Now applying this theorem, we may write $f_1(x) = \frac{f'(x)}{(1+2x^2)} = 0$, i.e. $f'(x) = 0$. This shows that $f(x)$ must be a constant, say $f(x) = c$. Let $c \neq 0$. As a matter of fact, from the hypothesis that $\int_{\Re} w(x)f(x)q_0(x)dx = 0$, i.e. $c\int_{\Re} \frac{e^{-x^2}}{(1+2x^2)^2} = 0 \Rightarrow \int_{\Re} \frac{e^{-x^2}}{(1+2x^2)^2} = 0$. This is not possible at all and consequently $c = 0$. Therefore $f(x) = 0$ is the only function for which $< f(x), q_n(x) >= 0$ always everywhere in $\Re$. It suffices to prove the completeness of our EOP's in $C_c^\infty(\Re)$. In fact we also know that $C_c^\infty(\Re)$ is dense in $\mathcal{L}^2(\Re)$. We know also $C_c^\infty(\Re)$ is always a subset of $\mathcal{L}^p$ for all possible values of p. And consequently, it is also admissible that the inner product of $f(x)$ with each element of closure of polynomial sequence $\{q_n\}_{n=0}^\infty$ is zero and hence we are able to satisfy the completeness of EOP sequence in the entire Hilbert space with certain weight. In other words, we conclude that the set of polynomials called EOP families is complete in the weighted Hilbert space $\mathcal{L}^2[\Re, w(x)]$, where $w(x)$ is the measure (or associated weight). However, we have already shown in [2] that those EOP's are orthogonal in $\mathcal{L}^2[\Re, w(x)]$.

3. Completeness of the EOP Families Associated to the Quantum System with Broken Supersymmetry

We studied above the completeness of EOP's associated to the conditionally exactly solvable systems (CES) and this system belongs to the category of unbroken supersymmetry. Now we shall consider another model associated with CES potentials which are supersymmetric (SUSY) partner of radial harmonic oscillator belonging to the category of broken supersymmetry and show that EOP families associated to that solvable quantum mechanical system are complete in weighted Hilbert space. In particular, we shall confine ourselves to the supersymmetric spherical oscillator so that the radial equation may be considered as one dimensional Schrödinger equation on the positive half line. More precisely, here radial harmonic oscillator is considered to be isotonic oscillator [16] because of the fact that it consists of a harmonic oscillator potential term as well as an additional rational function (centripetal barrier potential $\frac{\gamma(\gamma+1)}{2r^2}$) in which case γ denotes the orbital angular momentum quantum number. It is important to note that the centripetal barrier potential here does not make any physical sense in one dimension since the term $\frac{\gamma(\gamma+1)}{2x^2}$ singularities are often related to the radial equation for the three-dimensional harmonic oscillator. In this case the superpotential is given by ([17])

$$W(r) = r + \frac{\gamma+1}{r} + \frac{u'}{u}, \qquad 0 < r < \infty, \qquad\qquad \gamma \geq 0, \tag{5}$$

where $u(r^2)$ is given by

$$u(r^2) = {}_1F_1\left(\frac{1-\epsilon}{2}, \gamma + \frac{3}{2}, -r^2\right). \tag{6}$$

It can be easily verified that neither of $\psi_0^{\pm} = (u)^{\pm 1} r^{\pm(\gamma+1)} e^{\pm\frac{r^2}{2}}$ is normalizable so that the supersymmetry is broken. In this case the partner potentials are given by [2]

$$V_+(r) = \frac{r^2}{2} + \frac{\gamma(\gamma+1)}{2r^2} + \epsilon + \gamma + \frac{1}{2}, \tag{7}$$

$$V_-(r) = \frac{r^2}{2} + \frac{(\gamma+1)(\gamma+2)}{2r^2} + \frac{u'(r^2)}{u(r^2)}\left(2r + 2\frac{\gamma+1}{r} + \frac{u'(r^2)}{u(r^2)}\right) - \epsilon + \gamma + \frac{3}{2}. \tag{8}$$

The potential in (7) represents the standard radial oscillator potential whose energy and eigenfunctions are given by

$$E_n^+ = 2n+2\gamma+2+\epsilon, \quad \psi_n^+ = \sqrt{\frac{2(n!)}{\Gamma(n+\gamma+\frac{3}{2})}}\, r^{\gamma+1} L_n^{\gamma+\frac{1}{2}}(r^2)\, e^{-\frac{r^2}{2}}, \;\; n = 0, 1, 2, ... \tag{9}$$

Note that in this case V_- is a non shape invariant potential (or more precisely a conditionally exactly solvable one [18]) and has the same spectrum as V_+. Its eigenfunctions may be obtained using formalism of broken supersymmetry in Quantum mechanics [17, 19] in the form [2]

$$\psi_n^-(r) = \sqrt{\frac{2(n)!}{(4n+4\gamma+4+2\epsilon)\Gamma(n+\gamma+\frac{3}{2})}}$$

$$\times \frac{e^{-\frac{r^2}{2}} r^{\gamma+2}}{u(r^2)}\left[\frac{u'(r^2)}{r} L_n^{\gamma+\frac{1}{2}}(r^2) + 2u(r^2) L_n^{\gamma+\frac{3}{2}}(r^2)\right]. \tag{10}$$

We identify the expression inside the square bracket as the EOP $P_n(r^2)$ and the prefactor as the square root of the weight function $w(r^2)$, i.e.,

$$P_n(r^2) = \left[\frac{u'(r^2)}{r} L_n^{\gamma+\frac{1}{2}}(r^2) + 2u(r^2) L_n^{\gamma+\frac{3}{2}}(r^2)\right],$$

$$u(r^2) = {}_1F_1(\frac{1-\epsilon}{2}, \gamma + \frac{3}{2}, r^2) = \frac{2r^2 + 2\gamma + 3}{2\gamma + 3} \tag{11}$$

for $\epsilon = 3$ and now our focus is to show the completeness of those Exceptional orthogonal polynomial families $P_n(r^2)$ and this is the another set of EOP families. n is the nonnegative integer. In order to prove the completeness, first of all as before, we assume arbitrary

$$h(x) \in \mathcal{L}^2([0,\infty), w), \text{ where } w = \frac{(2\gamma+3)^2 e^{-x} x^{(\gamma+2)}}{(2x+2\gamma+3)^2},$$

$x = r^2 \in [0, \infty)$. It is to be noted that the new Exceptional orthogonal polynomials sequence $\{P_n\}_{n=0}^{\infty}$ will be complete iff $h(x) = 0$. Now,

$$P_n(x) = \frac{2}{2\gamma+3}\left[(2x+2\gamma+3)L_n^{\gamma+\frac{3}{2}}(x) + 2L_n^{\gamma+\frac{1}{2}}(x)\right]$$

$$= \frac{2}{2\gamma+3}\left[(2x+2\gamma+3)L_0^{\gamma+\frac{1}{2}}(x) + (2x+2\gamma+3)L_1^{\gamma+\frac{1}{2}}(x) + \right.$$

$$\left. ... + (2x+2\gamma+5)L_n^{\gamma+\frac{1}{2}}(x)\right].$$

Here we have used the relation $[L_n^{\alpha+1}(x) = \sum_{i=0}^{n} L_i^{\alpha}(x)]$.

It can be shown that

$$P_{n+1}(x) - P_n(x) = \frac{2}{2\gamma+3}\left[(2x+2\gamma+5)L_{n+1}^{\gamma+\frac{1}{2}}(x) - 2L_n^{\gamma+\frac{1}{2}}(x)\right].$$

Suppose
$< h(x), P_n(x) >= 0 \quad \forall n \implies \int_0^\infty g(x)(2x+2\gamma+5).1.e^{-x}x^{\gamma+\frac{1}{2}}dx = 0$, i.e. $< g(x)(2x+2\gamma+5), 1 >= 0$ and thus
$g(x) = \dfrac{h(x)x^{3/2}}{(2x+2\gamma+3)^2} \epsilon\ L^2(e^{-x}x^{\gamma+\frac{1}{2}}) = 0$, hence $h(x) = 0$. Consider the condition for $n \geq 1$.

Now $< h(x), P_n(x) >= 0$ and $< h(x), P_{n+1}(x) >= 0$ together implying $< h(x), P_{n+1}(x) - P_n(x) >= 0$, i.e. $\displaystyle\int_0^\infty \frac{h(x)x^{3/2}(2x+2\gamma+5)L_{n+1}^{\gamma+\frac{1}{2}}e^{-x}x^{\gamma+\frac{1}{2}}}{(2x+2\gamma+3)^2}dx$
$\displaystyle = 2\int_0^\infty \frac{h(x)x^{3/2}L_n^{\gamma+\frac{1}{2}}(x)e^{-x}x^{\gamma+\frac{1}{2}}}{(2x+2\gamma+3)^2}dx$. Hence

$$< g(x)(2x+2\gamma+5), L_{n+1}^{\gamma+\frac{1}{2}}(x) >_{w_{(\gamma+\frac{1}{2})}} = 2 < g(x), L_n^{\gamma+\frac{1}{2}}(x) >_{w_{(\gamma+\frac{1}{2})}}. \tag{12}$$

Moreover, $< g(x)(2x+2\gamma+5), g(x)(2x+2\gamma+5) >=\| g(x)(2x+2\gamma+5) \|^2$
$= \sum_{n=0}^{\infty} |< g(x)(2x+2\gamma+5), \frac{L_n^{\gamma+\frac{1}{2}}(x)}{\sqrt{c_n}} >|^2$
$= \sum_{n=1}^{\infty} |< g(x)(2x+2\gamma+5), \frac{L_n^{\gamma+\frac{1}{2}}(x))}{\sqrt{c_n}} >|^2.$

Also, $\| g(x) \|^2 = \sum_0^{\infty} |< g(x), \frac{L_n^{\gamma+\frac{1}{2}}(x)}{\sqrt{c_n}} >|^2.$

From (14) it is found that

$$c_{n+1} |< g(x)(2x+2\gamma+5), \frac{L_{n+1}^{\gamma+\frac{1}{2}}(x)}{\sqrt{c_{n+1}}} >|^2 = 4c_n |< g(x), \frac{L_n^{\gamma+1/2}(x)}{\sqrt{c_n}} >|^2, \quad (13)$$

i.e.

$$c_{n+1} |< g(x)(x+\gamma+5/2), \frac{L_{n+1}^{\gamma+1/2}(x)}{\sqrt{c_{n+1}}} >|^2 = c_n |< g(x), \frac{L_n^{\gamma+1/2}(x)}{\sqrt{c_n}} >|^2, \quad (14)$$

$c_n = \frac{\Gamma(n+\gamma+\frac{3}{2})}{n!}$ and $c_{n+1} = \frac{\Gamma(n+\gamma+\frac{5}{2})}{(n+1)!}$, since

$$\int_0^{\infty} x^{\alpha} e^{-x} L_n^{\alpha}(x) L_m^{\alpha}(x) dx = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{n,m}.$$

As $c_{n+1} > c_n \ \forall n \geq 0$ we have from (14)
$|< g(x)(x+\gamma+5/2), \frac{L_{n+1}^{\gamma+1/2}(x)}{\sqrt{c_{n+1}}} |^2 < \ |< g(x), \frac{L_n^{\gamma+1/2}(x)}{\sqrt{c_n}} >|^2$ and hence

$$\| g(x)(x+\gamma+\frac{5}{2}) \| < \| g(x) \| .$$

Thus we arrive at the comparison of $\mathcal{L}^2$ norms of the functions $g(x)(x+\gamma+\frac{5}{2})$ and $g(x)$ and this is not possible at all. So the only possibility is $g(x) = 0$, i.e. $h(x) = 0$ always everywhere in $[0,\infty)$ and hence, it suffices to prove the completeness of EOPs in connection with the broken supersymmetry in $\mathcal{L}^2[[0,\infty), w(x)]$. It is important to note that in [2] it has been shown those category of Exceptional orthogonal polynomials (EOP's) are orthogonal in $\mathcal{L}^2[[0,\infty), w(x)]$, where

$$w(x) = \frac{(2\gamma+3)^2 e^{-x} x^{(\gamma+2)}}{(2x+2\gamma+3)^2}, \quad x = r^2 \in [0,\infty).$$

4. Conclusion

We studied two exactly solvable quantum mechanical systems whose bound state wave function solutions are given in terms of Exceptional orthogonal polynomials (EOPs), as in [2]. We have applied new techniques to show completeness of two separate EOP families related to two distinct conditionally exactly solvable quantum systems in weighted $\mathcal{L}^2$ space. It would also be interesting to examine completeness of other EOP sequences associated to some other solvable quantum models.

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