International Journal of Applied Mathematics

Volume 27 No. 1 2014, 13-20
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v27i1.2

A DUAL HOMOLOGICAL INVARIANT AND SOME PROPERTIES

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Abstract: Based on the cohomology theory of groups, Andrade and Fanti defined in [1] an algebraic invariant, denoted by $E(G, \mathcal{S}, M)$, where G is a group, $\mathcal{S}$ is a family of subgroups of G with infinite index and M is a $\mathbb{Z}_2G$-module. In this work, by using the homology theory of groups instead of cohomology theory, we define an invariant "dual" to $E(G, \mathcal{S}, M)$, which we denote by $E_*(G, \mathcal{S}, M)$. The purpose of this paper is, through the invariant $E_*(G, \mathcal{S}, M)$, to obtain some results and applications in the theory of duality groups and group pairs, similar to those shown in Andrade and Fanti [2], and thus, providing an alternative way to get applications and properties of this theory.

AMS Subject Classification: 20E06, 05C25, 20J06
Key Words: homology of groups, duality, cohomological invariants

1. Introduction

Let G be a group and consider the homology and cohomology groups of G with coefficients in a $\mathbb{Z}_2G$-module M, denoted respectively by $H_*(G; M)$ and $H^*(G; M)$ (see Brown [7]).

Based on the cohomology theory of groups, Andrade and Fanti defined in

Received: November 15, 2013

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[1], a cohomological invariant denoted by $E(G, \mathcal{S}, M)$, where $\mathcal{S} = \{S_i, i \in I\}$ is a family of subgroups of G with $[G : S_i] = \infty$, $\forall\, i \in I$. Namely,

$$E(G, \mathcal{S}, M) = 1 + dim\ Ker\ res^G_{\mathcal{S}},$$

where $res^G_{\mathcal{S}} : H^1(G; M) \longrightarrow \prod_{i \in I} H^1(S_i; M)$ is the map induced by the inclusions $S_i \hookrightarrow G,\ i \in I$.

By means of this invariant results in duality theory and splitting of groups were proved in Andrade and Fanti in [1], [2] and Andrade et al. in [3], [4].

In this work, we define define a "dual" invariant $E_*(G, \mathcal{S}, M)$, by using the theory of homology of groups, and we establish some results, similar to those stated in Andrade and Fanti [2], in the theory of duality groups.

2. Definition of the Invariant $E_*(G, \mathcal{S}, M)$ and Some Properties

Let G be a group, $\mathcal{S} = \{S_i, i \in I\}$ a family of subgroups of G and M a $\mathbb{Z}_2G$-module. Denote $\bigoplus_{i \in I} H_*(S_i; M)$ by $H_*(\mathcal{S}; M)$. From Bieri and Eckmann [6], Prop. 1.1, we have the long exact sequence:

$$\cdots \to H_1(\mathcal{S}; M) \overset{cor^G_{\mathcal{S}}}{\to} H_1(G; M) \overset{J}{\to} H_1(G, \mathcal{S}; M) \overset{\delta}{\to}$$
$$\overset{\delta}{\to} H_0(\mathcal{S}; M) \overset{cor_{0,\mathcal{S}}^G}{\to} H_0(G; M) \to 0. \quad (2.1)$$

Definition 1. Let $(G, \mathcal{S})$ be a group pair, $\mathcal{S} = \{S_i, i \in I\}$ a family of subgroups of G with $[G : S_i] = \infty$, $\forall\, i \in I$, and M a $\mathbb{Z}_2G$-module. We define:

$$E_*(G, \mathcal{S}, M) = 1 + dim\ coker\ cor^G_{\mathcal{S}},$$

where $cor^G_{\mathcal{S}} : \bigoplus_{i \in I} H_1(S_i; M) \longrightarrow H_1(G; M)$ is the homomorphism that appears in the long exact sequence (2.1), which is induced in homology by the inclusions $S_i \overset{i}{\hookrightarrow} G$.

Remark 1. Observe that, by the long exact sequence (2.1), we have:

$$\text{coker } cor^G_{\mathcal{S}} = \frac{H_1(G; M)}{\text{Im } cor^G_{\mathcal{S}}} = \frac{H_1(G; M)}{\ker J} \simeq \text{Im } J.$$

Then, we can define the invariant E_* by:

$$E_*(G, \mathcal{S}, M) = 1 + \dim \mathrm{Im}\ \ J.$$

Let $\mathcal{C}$ the category whose objects are pairs $((G, \mathcal{S}); M)$ where G is a group, $\mathcal{S} = \{S_i,\ i \in I\}$ is a family of subgroups of G and M is a $\mathbb{Z}_2G$-module, and whose morphisms are maps:

$$\psi : ((G, \mathcal{S}); M) \longrightarrow ((L, \mathcal{R} = \{R_j, j \in J\}); N)$$

consisting of:

(a) a homomorphism of groups $\alpha : G \longrightarrow L$;

(b) a map $\pi : I \longrightarrow J$ such that $\alpha(S_i) \subset R_{\pi(i)}$;

(c) a map $\phi : M \longrightarrow N$ such that $\phi(g.m) = \alpha(g).\phi(m)$, i.e., ϕ is a $\mathbb{Z}_2G$-homomorphism via $\alpha : G \longrightarrow L$.

More precisely, if ψ is an isomorphism in $\mathcal{C}$, α is an isomorphism of groups, π is a bijection and ϕ is a $\mathbb{Z}_2G$-isomorphism.

We will prove now that $E_*(G, \mathcal{S}, M)$ is an invariant in the category $\mathcal{C}$.

Theorem 1. *If $((G, \mathcal{S}); M)$ and $((L, \mathcal{R}); N)$ are isomorphic in the category $\mathcal{C}$, then*

$$E_*(G, \mathcal{S}, M) = E_*(L, \mathcal{R}, N).$$

Proof: Consider the isomorphism $\rho : \mathbb{Z}_2(G/\mathcal{S}) \longrightarrow \mathbb{Z}_2(L/\mathcal{R})$ defined on the generators by $\rho(xS_i) = \alpha(x)R_{\pi(i)}$, whose inverse $\beta : \mathbb{Z}_2(L/\mathcal{R}) \longrightarrow \mathbb{Z}_2(G/\mathcal{S})$ is given by $\beta(lR_j) = \alpha^{-1}(l)S_{\pi^{-1}(j)}$. As $\rho(\Delta) = \Delta'$, it follows that the restriction map $\bar{\rho} : \Delta \longrightarrow \Delta'$ is an isomorphism and we obtain the commutative diagram with exact lines:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Delta & \xrightarrow{i_G} & \mathbb{Z}_2(G/\mathcal{S}) & \xrightarrow{\varepsilon_G} & \mathbb{Z}_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar{\rho}} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & \Delta' & \xrightarrow{i_L} & \mathbb{Z}_2(L/\mathcal{R}) & \xrightarrow{\varepsilon_L} & \mathbb{Z}_2 & \longrightarrow & 0
\end{array}$$

Since $\mathbb{Z}_2$ is a $\mathbb{Z}_2$-projective module, the maps ε_G and ε_L have right inverse. By Hu [8], (I.5.10), i_G e i_L have left inverse. Therefore, the above diagram induces the following commutative diagram with exact lines:

$$\begin{array}{cccccccccc}
0 & \longrightarrow & \Delta \otimes M & \longrightarrow & \mathbb{Z}_2(G/\mathcal{S}) \otimes M & \longrightarrow & \mathbb{Z}_2 \otimes M & \longrightarrow & 0 & (1) \\
 & & \downarrow{\scriptstyle \bar{\rho}\otimes\phi} & & \downarrow{\scriptstyle \rho\otimes\phi} & & \downarrow{\scriptstyle id\otimes\phi} & & & \\
0 & \longrightarrow & \Delta' \otimes N & \longrightarrow & \mathbb{Z}_2(L/\mathcal{R}) \otimes N & \longrightarrow & \mathbb{Z}_2 \otimes N & \longrightarrow & 0 & (2)
\end{array}$$

Denote $\mathcal{M} = \Delta \otimes M$, $\mathcal{N} = \Delta' \otimes N$, $\bar{\rho} \otimes \phi$ by $\bar{\phi}$ and remember that $\mathbb{Z}_2 \otimes M \simeq M$. We have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathbb{Z}_2(G/\mathcal{S}) \otimes M & \longrightarrow & M & \longrightarrow & 0 \quad (1) \\
 & & \downarrow \bar{\rho} & & \downarrow \rho\otimes\phi & & \downarrow \phi & & \\
0 & \longrightarrow & \mathcal{N} & \longrightarrow & \mathbb{Z}_2(L/\mathcal{R}) \otimes N & \longrightarrow & N & \longrightarrow & 0 \quad (2)
\end{array}$$

The horizontal maps of the sequence (1) are $\mathbb{Z}_2G$-homomorphisms and the maps of the sequence (2) are $\mathbb{Z}_2L$-homomorphisms. The vertical maps are $\mathbb{Z}_2G$-isomorphisms, considering the $\mathbb{Z}_2L$-modules as $\mathbb{Z}_2G$-modules via α. Let $P \twoheadrightarrow \mathbb{Z}_2$ be a projective resolution of $\mathbb{Z}_2$ on $\mathbb{Z}_2L$. Then, $P \twoheadrightarrow \mathbb{Z}_2$ is also a projective resolution of $\mathbb{Z}_2$ on $\mathbb{Z}_2G$. In fact, $\alpha : G \longrightarrow L$ induces the following action of G on P_i : $g * x = \alpha(g).x$, $\forall\ g \in G$ and $x \in P_i$. Thus, each P_i is a $\mathbb{Z}_2G$-module. Since α is an isomorphism, each P_i is $\mathbb{Z}_2G$-projective. Let $\tau : P \longrightarrow P$ be the identity map. We have that τ is compatible with α, i.e., $\tau(g * x) = \alpha(g)\tau(x)$. Applying $P \otimes_{\mathbb{Z}_2G}$ __ on the sequence (1) and $P \otimes_{\mathbb{Z}_2L}$ __ on the sequence (2), we have the following chain complex commutative diagram:

$$\begin{array}{ccccccccc}
0 \to & P \otimes_{\mathbb{Z}_2G} \mathcal{M} & \to & P \otimes_{\mathbb{Z}_2G} (\mathbb{Z}_2(G/\mathcal{S}) \otimes M) & \to & P \otimes_{\mathbb{Z}_2G} M & \to 0 & (1') \\
 & \downarrow \tau\otimes\bar{\phi} & & \downarrow \tau\otimes(\rho\otimes\phi) & & \downarrow \tau\otimes\phi & & \\
0 \to & P \otimes_{\mathbb{Z}_2L} \mathcal{N} & \to & P \otimes_{\mathbb{Z}_2L} (\mathbb{Z}_2(L/\mathcal{R}) \otimes N) & \to & P \otimes_{\mathbb{Z}_2L} N & \to 0 & (2')
\end{array}$$

The sequences (1') e (2') are exact because $P \twoheadrightarrow \mathbb{Z}_2$ is a $\mathbb{Z}_2G$-projective resolution and also a $\mathbb{Z}_2L$-projective resolution. Furthermore, the vertical maps are isomorphisms. Applying the functor $H_*($__$)$, the Shapiro Lemma and the definition of relative homology, we have the following commutative diagram with exact lines (Brown [7], I.0.4)

$$\begin{array}{ccccccccccc}
\cdots \to & H_1(\mathcal{S};M) & \xrightarrow{cor_{\mathcal{S}}^G} & H_1(G;M) & \xrightarrow{J_G} & H_1(G,\mathcal{S};M) & \xrightarrow{\delta_G} & H_0(\mathcal{S};M) & \xrightarrow{cor_{0,\mathcal{S}}^{G}} & H_0(G;M) & \to 0 \\
 & \downarrow \simeq & & \downarrow \psi \simeq & & \downarrow \gamma \simeq & & \downarrow \simeq & & \downarrow \simeq & \\
\cdots \to & H_1(\mathcal{R};N) & \xrightarrow{cor_{\mathcal{R}}^L} & H_1(L;N) & \xrightarrow{J_L} & H_1(L,\mathcal{R};N) & \xrightarrow{\delta_L} & H_0(\mathcal{R};N) & \xrightarrow{cor_{0,\mathcal{R}}^{L}} & H_0(L;N) & \to 0
\end{array}$$

The vertical maps are isomorphisms because are induced by isomorphisms. Furthermore, as $\ker J_G \overset{\psi}{\simeq} \ker J_L$, it follows that the induced map

$$\bar{\psi} : \frac{H_1(G;M)}{\ker J_G} \longrightarrow \frac{H_1(L,N)}{\ker J_L}$$

is an isomorphism. Then, $E_*(G,\mathcal{S};M) = 1 + \dim(\text{coker } cor_{\mathcal{S}}^{G}) = 1 + \dim \dfrac{H_1(G;M)}{\text{Im } cor_{\mathcal{S}}^{G}} = 1 + \dim \dfrac{H_1(G;M)}{\ker J_G} = 1 + \dim \dfrac{H_1(L;N)}{\ker J_L} = 1 + \dim \dfrac{H_1(L;N)}{\text{Im } cor_{\mathcal{R}}^{L}} = 1 + \dim(\text{coker } cor_{\mathcal{R}}^{L}) = E_*(L,\mathcal{R};N)$. □

We present below a characterization of the invariant $E_*(G,\mathcal{S},M)$ when the involved spaces have finite dimension.

Proposition 1. *Let $(G,\mathcal{S})$ be a group pair where $\mathcal{S} = \{S_i, i \in I\}$ with $[G : S_i] = \infty$, $\forall\ i \in I$, and M is a $\mathbb{Z}_2G$-module. If the vectorial spaces $H_0(G;M)$, $H_0(\mathcal{S};M) := \bigoplus_{i\in I} H_0(S_i;M)$ and $H_1(G,\mathcal{S};M)$ have finite dimension, then:*

$$E_*(G,\mathcal{S},M) = 1 + dim\ H_0(G;M) - dim\ H_0(\mathcal{S};M) + dim\ H_1(G,\mathcal{S};M).$$

Proof: The long exact sequence (2.1) provides the following short exact sequence:

$$0 \longrightarrow Im\ J \stackrel{\chi}{\longrightarrow} H_1(G,\mathcal{S};M) \stackrel{\delta}{\longrightarrow} H_0(\mathcal{S};M) \stackrel{cor_{0},{}_{\mathcal{S}}^{G}}{\twoheadrightarrow} H_0(G;M) \longrightarrow 0,$$

where χ denotes the inclusion map. Since the vectorial spaces have finite dimension, we have:

$$\begin{aligned}
\dim H_0(G;M) &= \dim \frac{H_0(\mathcal{S};M)}{\ker cor_0,{}_{\mathcal{S}}^{G}} \\
&= \dim\ H_0(\mathcal{S};M) - \dim \ker cor_0,{}_{\mathcal{S}}^{G} \\
&= \dim H_0(\mathcal{S};M) - \dim \text{Im}\ \ \delta \\
&= \dim H_0(\mathcal{S};M) - \dim \tfrac{H_1(G,\mathcal{S};M)}{\ker \delta} \\
&= \dim H_0(\mathcal{S};M) - [\dim H_1(G,\mathcal{S};M) - \dim \ker \delta] \\
&= \dim\ H_0(\mathcal{S};M) - \dim H_1(G,\mathcal{S};M) + \dim \text{Im}\ \ J.
\end{aligned}$$

Thus, we have $E_*(G,\mathcal{S},M) = \dim ImJ = \dim H_0(G;M) - \dim H_0(\mathcal{S};M) + \dim H_1(G,\mathcal{S};M)$. □

3. The Invariant $E'_*(G,\mathcal{S})$ and Duality

In this section we study the dual invariant $E_*(G,\mathcal{S},M)$ when the $\mathbb{Z}_2G$-module M is the trivial module $\mathbb{Z}_2$. We will denote $E_*(G,\mathcal{S},\mathbb{Z}_2)$ by

$E'_*(G, \mathcal{S})$.

Proposition 2. *Let G be a group and $\mathcal{S} = \{S\}$ a family with only one subgroup of G and $[G : S] = \infty$. Then:*

$$E_*(G, S) = 1 + dim\ H_1(G, S; \mathbb{Z}_2).$$

Proof: Consider the exact sequence:

$$\cdots \longrightarrow H_1(S; \mathbb{Z}_2) \xrightarrow{cor_S^G} H_1(G; \mathbb{Z}_2) \xrightarrow{J} H_1(G, S; \mathbb{Z}_2) \xrightarrow{\delta}$$

$$\xrightarrow{\delta} H_0(S; \mathbb{Z}_2) \xrightarrow{cor_0,_S^G} H_0(G; \mathbb{Z}_2) \longrightarrow 0.$$

Since $H_0(G; \mathbb{Z}_2) = (\mathbb{Z}_2)_G \simeq \mathbb{Z}_2$, $H_0(S; \mathbb{Z}_2) = (\mathbb{Z}_2)_S \simeq \mathbb{Z}_2$ and $cor_0,_S^G$ is surjective, it follows that $cor_0,_S^G$ is isomorphism, and so, the long exact sequence above becomes

$$\cdots \longrightarrow H_1(S; \mathbb{Z}_2) \xrightarrow{cor_S^G} H_1(G; \mathbb{Z}_2) \xrightarrow{J} H_1(G, S; \mathbb{Z}_2) \longrightarrow 0.$$

Furthermore, Im $J = H_1(G, S; \mathbb{Z}_2)$. Then, by Remark 1, we have

$$E'_*(G, S) = 1 + \dim H_1(G, S; \mathbb{Z}_2).$$

□

Example 1. Let $G = \langle a \rangle \oplus \langle b \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}$ and $\mathcal{S} = \{S\}$, $S = \langle a \rangle \simeq \mathbb{Z}$, where $a = [\alpha]$ and $b = [\beta]$. Consider X the torus T^2 and Y the loop α. We have that (X, Y) is an Eilenberg-MacLane pair realizing (G, S). By Bieri and Eckmann [6], Theorem 1.3, $H_1(G, S; \mathbb{Z}_2) = H_1(T^2, S^1; \mathbb{Z}_2) = \mathbb{Z}_2$. Then: $E'_*(G, S) = 2$.

When (G, S) is a Poincaré duality pair with $[G : S] = \infty$ we have the following results:

Proposition 3. *If (G, S) is a PD^n-pair with $[G : S] = \infty$, then*

$$E'_*(G, S) = 1 + dim\ H^{n-1}(G; \mathbb{Z}_2).$$

Proof: Since (G, S) is a PD^n-pair, we have:

$$H_1(G, S; \mathbb{Z}_2) \simeq H_1(G; \mathbb{Z}_2 \otimes_{\mathbb{Z}_2} \mathbb{Z}_2) = H^{n-1}(G; \mathbb{Z}_2).$$

By Proposition 2, we have $E'_*(G, S) = 1 + dim\ H^{n-1}(G; \mathbb{Z}_2)$. □

Corollary 1. *If (G, S) is a PD^n-pair with $[G : S] = \infty$, then*

$$E'_*(G, S) = 1 + dim\ \Delta_G,$$

where $\varepsilon : \mathbb{Z}_2(G/S) \longrightarrow \mathbb{Z}_2$ is the augmentation map and $\Delta = Ker(\varepsilon)$.

Proof: We have that G is a duality group with dualizing module Δ. Then,

$$H^{n-1}(G; \mathbb{Z}_2) \simeq H_0(G; \Delta) = \Delta_G.$$

Therefore, $E'_*(G, S) = 1 + dim\ \Delta_G$. □

Example 2. Consider $G = \langle a \rangle * \langle b \rangle$ and $\mathcal{S} = \{S\}$, $S = \langle bab^{-1}a^{-1} \rangle$ and let X be a torus from which an open disk has been removed and $Y = \partial X \simeq S^1$. Hence, X is homotopy equivalent to the figure eight. Thus $G \simeq \pi_1(X) = \langle a \rangle * \langle b \rangle = \mathbb{Z} * \mathbb{Z}$, $S \simeq \pi_1(Y) = \langle bab^{-1}a^{-1} \rangle$ and (X, Y) is an Eilenberg-MacLane pair realizing (G, S). It follows that (G, S) is a PD^2-pair and $H^1(G; \mathbb{Z}_2) = H^1(X; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then,

$$E'_*(G, S) = 1 + \dim H^1(G; \mathbb{Z}_2) = 1 + \dim H^1(X; \mathbb{Z}_2) = 3.$$

Theorem 2. *Let $(G, \mathcal{S})$ be a PD^n-pair, $\mathcal{S} = \{S_i, i = 1, ..., r\}$, with $[G : S_i] = \infty$, $\forall\ i = 1, ..., r$ and G finitely presented. Then*

$$E'_*(G, \mathcal{S}) = 2 - r + dim\ H^{n-1}(G; \mathbb{Z}_2)\ <\ \infty.$$

Proof: If $(G, \mathcal{S})$ is a PD^n-pair, $\mathcal{S} = \{S_i, i = 1, ..., r\}$, we have that G is a D^{n-1}-group and consequently, by Bieri [5], Theorem 9.2, G is of type FP. Since G is finitely presented there is a finitely dominated $K(G, 1)$-complex (Brown [7], VIII.7.1). Thus, we have that: $H_0(G; \mathbb{Z}_2) \simeq \mathbb{Z}_2$, $H_0(\mathcal{S}; \mathbb{Z}_2) = \prod_{i=1}^{r} H_0(S_i; \mathbb{Z}_2) \simeq \prod_{i=1}^{r} \mathbb{Z}_2$ and $H_1(G, \mathcal{S}; \mathbb{Z}_2) \simeq H^{n-1}(G; \mathbb{Z}_2)$. Since $dim\ H^{n-1}(G; \mathbb{Z}_2) < \infty$, we have, by Proposition 1,
$E'_*(G, \mathcal{S}) = 1 + dim\ H_0(G; M) - dim\ H_0(\mathcal{S}; M) + dim\ H_1(G, \mathcal{S}; M) = 1 + 1 - r + dim\ H^{n-1}(G; \mathbb{Z}_2) = 2 - r + dim\ H^{n-1}(G; \mathbb{Z}_2) < \infty$. □

Corollary 2. *If $(G, \mathcal{S})$, $\mathcal{S} = \{S_i, i = 1, ..., r\}$, is a PD^n-pair, with G finitely presented and $[G : S_i] = \infty$, $\forall\ i = 1, ..., r$, then*

$$r \leq 1 + dim\ H^{n-1}(G; \mathbb{Z}_2) < \infty.$$

Proof: $E'_*(G, \mathcal{S}) \geq 1 \Longrightarrow 1 \leq 2 - r + dim\ H^{n-1}(G; \mathbb{Z}_2) < \infty \Longrightarrow r \leq 1 - dim\ H^{n-1}(G; \mathbb{Z}_2) < \infty$. □

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