International Journal of Applied Mathematics

Volume 27 No. 2 2014, 137-145
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v27i2.3

OSCILLATION AND NON OSCILLATION FOR THE SOLUTIONS OF CERTAIN TYPE OF GENERALIZED NEUTRAL α-DIFFERENCE EQUATION

M. Maria Susai Manuel[1 §], K. Srinivasan[2]
V. Chandrasekar[3], G. Dominic Babu[4]

[1]Department of Science and Humanities
R.M.D. Engineering College
Kavaraipettai, 601 206, Tamil Nadu, S. INDIA
[2]Department of Mathematics
S.K.P. Institute of Technology
Tiruvannamalai, 606 611, S. INDIA
[3]Department of Science and Humanities
S.K.P. Engineering College
Tiruvannamalai, Tamil Nadu, S. INDIA
[4]Department of Mathematics
Sacred Heart College
Tirupattur, 635 601, Vellore District, Tamil Nadu, S. INDIA

Abstract: In this paper, the authors discuss the oscillation and non oscillation for generalized neutral α-difference equation

$$\Delta_{\alpha(\ell)}(u(k) + pu(k - \tau\ell)) + q(k)u(k - \sigma\ell) = 0, \quad k \in [0, \infty), \tag{1}$$

where p is a constant, $q(k)$ is defined on $[0, \infty)$, τ is a positive integer and σ is a non-negative integer.

AMS Subject Classification: 39A10, 39A11, 39A13, 39A20
Key Words: generalized α-difference equation, oscillation and nonoscillation

Received: January 25, 2014

§Correspondence author

1. Introduction

The theory of difference equations is based on the operator Δ defined as

$$\Delta u(k) = u(k+1) - u(k), \qquad k \in \mathbb{N} = \{0, 1, 2, ...\}. \tag{2}$$

Even though many authors ([1],[9]) have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k), \qquad k \in [0, \infty),\ \ell \in (0, \infty), \tag{3}$$

no significant progress took place on this line. But recently M. Maria Susai Manuel, G.B.A. Xavier and E. Thandapani [3], took up the definition of Δ as given in (3), and developed the theory of difference equations in a different direction and many interesting results were obtained in number theory. For convenience, the authors labeled the operator Δ defined by (3) as Δ_ℓ and its inverse by Δ_ℓ^{-1}. When Δ_ℓ is operated on a complex function $u(k)$ and considering ℓ to be real, some new qualitative properties like rotatory, expanding, shrinking, spiral and weblike were noticed. The results obtained can be found in [3]-[7].

Jerzy Popenda [2], while discussing the behavior of solutions of a particular type of difference equation, defined Δ_α as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. This definition of Δ_α is being ignored for a long time. Recently, M. Maria Susai Manuel, V. Chandrasekar and G. Britto Antony Xavier [8] have generalized the definition of Δ_α by $\Delta_{\alpha(\ell)}$ defined as $\Delta_{\alpha(\ell)} u(k) = u(k+\ell) - \alpha u(k)$ for the real valued function $u(k)$ and $\ell \in (0, \infty)$ and also obtained the solutions of certain types of generalized α-difference equations, in particular, the generalized Clairaut's α-difference equation, generalized Euler α-difference equation and the generalized α-Bernoulli polynomial $B_{\alpha(n)}(k, \ell)$, which is a solution of the α-difference equation $u(k+\ell) - \alpha u(k) = nk^{n-1}$, for $n \in \mathbb{N}(1)$. In this paper, we present solutions of certain type of generalized neutral α-difference equations and discuss the oscillatory and non oscillatory behavior of generalized neutral α-difference equation (1).

Throughout this paper, we make use of the following assumptions:

$$\mu = \max\{\tau, \sigma\}.$$

Then by a solution of (1) we mean a function $u(k)$ which is defined for $k \geq -\mu$ and satisfies the equation (1) for $k \in [0, \infty)$. Clearly if

$$u(k) = A_k, \quad k \in [-\mu, 0] \tag{4}$$

are given, then (1) has a unique solution, and it can be constructed recursively. Also, assume that function $q(k)$ is not identically zero.

2. Preliminaries

In this section, we present the definition of the generalized α-difference equation of the n^{th} kind, from which the equation (1) becomes the generalized linear α-difference equation of the third kind by properly selecting the values of ℓ_i for $i = 1, 2, 3$.

Definition 2.1. Let $L = \{\ell_1, \ell_2, \cdots, \ell_n\}$ be a set of n positive real numbers, $r(L)$ be the set of all subsets of size r from the set L and $\alpha > 0$ be fixed. Then, for $k \in [0, \infty)$, we define the generalized n^{th} kind α-difference equation as

$$F\left(\left(k, \left(P_A(k,\alpha), u(k + \sum_{\ell_i \in A} a_i^A \ell_i)\right)_{A \in r(L)}\right)_{r=0}^{n}\right) = 0, \tag{5}$$

and the generalized n^{th} kind linear α-difference equation as

$$\sum_{r=0}^{n} \sum_{A \in r(L)} P_A(k,\alpha) u\Big(k + \sum_{\ell_i \in A} a_i^A \ell_i\Big) = f(k), \tag{6}$$

where $P_A(k,\alpha)$, $f(k)$ and F are real valued functions and a_i^A's are constants.

Remark 2.2. i) When $\ell_i = \ell$, for $i = 1, 2, \cdots, n$, the equation (5) (the equation (6)) becomes the generalized n^{th} order (linear) α-difference equation.

ii) When $\ell_i = 1$, for $i = 1, 2, \cdots, n$ and $k \in \mathbb{N}(a)$, a is an integer, the equation (5) (the equation (6)) becomes the n^{th} order (linear) α-difference equation.

iii) When $\ell_i = 1$, for $i = 1, 2, \cdots, n$, $\alpha = 1$ and $k \in \mathbb{N}(a)$, a is an integer, the equation (5) (the equation (6)) becomes the n^{th} order (linear) difference equation.

iv) Equation (5) (the equation (6)) becomes the Delay or Neutral type difference equation by taking $\ell_i = 1$, for $i = 1, 2, \cdots, n$, $\alpha = 1$, $k \in \mathbb{N}(a)$, a is an integer, negative values for certain a_i's.

The following example illustrates Equation (6).

Example 2.3. Equation (1) can be expressed as

$$-\alpha u(k) + u(k+\ell) - p\alpha u(k-\tau\ell) + q(k)u(k-\sigma\ell) + pu(k+\ell-\tau\ell) = 0.$$

By taking $\ell_1 = \ell$, $\ell_2 = \tau\ell$ and $\ell_3 = \sigma\ell$ we get $L = \{\ell, \tau\ell, \sigma\ell\}$,

$$0(L) = \{\phi\},\ 1(L) = \{\{\ell\}, \{\tau\ell\}, \{\sigma\ell\}\},$$
$$2(L) = \{\{\ell, \tau\ell\}, \{\ell, \sigma\ell\}, \{\tau\ell, \sigma\ell\}\} \text{ and } 3(L) = \{\{\ell, \tau\ell, \sigma\ell\}\}.$$

Now, if we take $P_{\{\phi\}}(k,\alpha) = -\alpha$, $P_{\{\ell\}}(k,\alpha) = 1$, $P_{\{\tau\ell\}}(k,\alpha) = -p\alpha$, $P_{\{\sigma\ell\}} = q(k)$, $P_{\{\ell,\tau\ell\}}(k,\alpha) = p$, $P_{\{\ell,\sigma\ell\}}(k,\alpha) = P_{\{\tau\ell,\sigma\ell\}}(k,\alpha) = 0$, $P_{\{\ell,\tau\ell,\sigma\ell\}}(k,\alpha) = 0$, $a_1^{\{\ell\}} = 1$, $a_2^{\{\tau\ell\}} = -1$, $a_3^{\{\sigma\ell\}} = -1$, $a_1^{\{\ell,\tau\ell\}} = -1$, $a_2^{\{\ell,\tau\ell\}} = 1$ and all other a_i^A's are zero in (6) then, equation (1) is a generalized third kind linear α-difference equation.

Definition 2.4. A nontrivial solution $u(k)$ of (1) is said to be oscillatory, if for every $k > 0 \in [0,\infty)$ there exists a $k \geq K$ such that $u(k)u(k+\ell) \leq 0$. The equation (1) itself is called oscillatory if all its solutions are oscillatory. Otherwise, it is called nonoscillatory.

3. Main Results

Lemma 3.1. *Let $\ell, \alpha > 0$ and $\alpha \neq 1$. If $v(k)$ is a solution of the generalized first order linear α-difference equation*

$$-\alpha v(k) + v(k+\ell) = u(k), \tag{7}$$

$$\text{then} \quad w(k) = v(k) - \alpha^{\left[\frac{k}{\ell}\right]} c_j, \tag{8}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$ is also a solution of (7).

Proof. Since (8) satisfies (7), the proof is obvious. ☐

Theorem 3.2. *Let $u(k)$ be defined for all $k \in [0,\infty)$. Then, for $k \in [\ell,\infty)$, $v(k) = \sum_{r=1}^{\left[\frac{k}{\ell}\right]} \alpha^{r-1} u(k - r\ell)$ is a solution of the generalized linear nonhomogeneous α-difference equation*

$$-\alpha v(k) + v(k+\ell) = u(k). \tag{9}$$

Proof. Replacing k by $k-\ell$ and $k-2\ell$ in (9), we find

$$v(k) = \alpha v(k-\ell) + u(k-\ell), \tag{10}$$

$$\text{and} \quad v(k-\ell) = \alpha v(k-2\ell) + u(k-2\ell), \tag{11}$$

which yield $\qquad v(k) = u(k-\ell) + \alpha u(k-2\ell) + \alpha^2 v(k-2\ell).$

The proof follows by repeating this process again and again. □

Theorem 3.3. *Let $\alpha \neq c^\ell$, $k \in [\ell, \infty)$ and $j = k - \left[\frac{k}{\ell}\right]\ell$. Then*

$$w(k) = \frac{kc^k}{(c^\ell - \alpha)} - \frac{\ell c^{k+\ell}}{(c^\ell - \alpha)^2} - \alpha^{\left[\frac{k}{\ell}\right]} c_j, \tag{12}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$ is a solution of the generalized first order linear α-difference equation

$$-\alpha v(k) + v(k+\ell) = kc^k. \tag{13}$$

Proof. The proof follows by taking $u(k) = kc^k$ in (7) and

$$v(k) = \frac{kc^k}{(c^\ell - \alpha)} - \frac{\ell c^{k+\ell}}{(c^\ell - \alpha)^2}$$

in (8). □

Lemma 3.4. *Let $q(k) \geq 0$ for all $k \in [0, \infty)$ and let $u(k)$ be an eventually positive solution of (1). Set $z(k) = u(k) + pu(k - \tau\ell)$.*

(a) If $p = -1$, then $z(k) > 0$ and $\Delta_{\alpha(\ell)} z(k) \leq 0$ eventually.

(b) If $-1 < p < 0$, then $z(k) > 0$ and $\Delta_{\alpha(\ell)} z(k) < 0$ eventually.

(c) If $p < -1$ and $\sum\limits_{k=1}^{\infty} p(k\ell + j) = \infty$, then $z(k) < 0$ and $\Delta_{\alpha(\ell)} z(k) \leq 0$ eventually.

Proof. Since $q(k) \neq 0$, from the equation (1), we have

$$\Delta_{\alpha(\ell)} z(k) = -q(k)u(k - \sigma\ell) \leq 0,$$

eventually, so $z(k)$ cannot be eventually identically zero. Thus, it follows that $z(k)$ is eventually positive or eventually negative.

If $z(k) < 0$ eventually, then $z(k) \leq z(K) < 0$ for $k \geq K \in [0, \infty)$. Hence

$$u(K + \tau k) \leq z(K) + u(K + (k - \ell)\tau) \leq \ldots \leq kz(K) + u(K).$$

On letting $k \to \infty$ in the above inequality, we find $u(K + \tau k)$ to be negative, which is a contradiction to $u(k) > 0$. This proves (a).

The proof of (b) is similar to that of (a).

To prove (c), again from (1), we have $\Delta_{\alpha(\ell)} z(k) = -q(k)u(k - \sigma\ell) \leq 0$, for all large k. We shall prove that $z(k) < 0$, eventually. If not, then $z(k) = u(k) + pu(k - \tau\ell) \geq 0, \quad k \geq K$,

$$\text{i.e.} \quad u(k) \geq -pu(k - \tau\ell), \quad k \geq K$$

which implies that

$$0 < u(K - \tau\ell) \leq \left(\frac{-1}{p}\right) u(K) \leq \ldots \leq \left(\frac{-1}{p}\right)^r u(K + (r-1)\tau\ell),$$

$r = 1, 2, \ldots$. On letting $r \to \infty$ in the above inequality, we get $u(k) \to \infty$ as $k \to \infty$. But, then since $\Delta_{\alpha(\ell)} z(k) = -q(k)u(k - \tau\ell) \leq -Lq(k)$ for large k, where L is a positive number. On summing the last inequality, we obtain

$$z(k + \ell) - \alpha z(k) \leq -L \sum_{r=1}^{\left[\frac{k}{\ell}\right]} \alpha^{r-1} q(k - r\ell),$$

which implies that $z(k) \to -\infty$ as $k \to \infty$. This contradicts the fact that $z(k) \geq 0$ for $k \geq K$. □

Now we shall establish oscillation criteria for the difference equation (1). The results obtained depend on the values of the parameter p.

Theorem 3.5. *Assume that $p = -1$, $q(k) \geq 0$ for $k \in \mathbb{N}(1)$, and for a positive integer K,*

$$\sum_{r=0}^{\infty} (\frac{1}{\alpha})^r q(K + r\ell) = \infty. \tag{14}$$

Then the equation (1) is oscillatory.

Proof. Assume the contrary. Without loss of generality let $u(k)$ be an eventually positive solution of (1). By Lemma (3.4), $z(k) = u(k) + pu(k - \tau\ell) > 0$ and $\Delta_{\alpha(\ell)} z(k) \leq 0$, eventually. This implies that $\lim_{k\to\infty} z(k) = \gamma \geq 0$ exists.

On summing (1) from K to k, we get

$$z(k + \ell) - \alpha z(k) + (1 - \alpha) \sum_{r=0}^{n-1} z(K + r\ell) + \sum_{r=0}^{n-1} (\frac{1}{\alpha})^r q(r)u(K + r\ell - \sigma) = 0.$$

On letting $k \to \infty$ in the above equation, we obtain

$$z(K) \geq \sum_{r=0}^{\infty} \big(\frac{1}{\alpha}\big)^r q(r) u(K + r\ell - \sigma). \tag{15}$$

Now setting $\min_{K<r<K+\tau\ell} u(K + r\ell - \tau\ell) = s > 0$, we find from $z(k) = u(k) - u(k - \tau\ell) > 0$ for $k \geq K$ that $u(k) \geq s$ for $k \geq K$. Thus, from (15,) we have

$$\infty > z(K + r\ell + \sigma\ell) \geq \sum_{r=0}^{\infty} \big(\frac{1}{\alpha}\big)^r q(k + r\ell + \sigma\ell) u(k + r\ell - \sigma\ell)$$

$$\geq \sum_{r=0}^{\infty} \big(\frac{1}{\alpha}\big)^r q(k + r\ell + \sigma\ell)$$

which contradicts condition (14). □

Example 3.6. Consider the difference equation

$$\Delta_{\alpha(\ell)}(u(k) - u(k - \tau\ell)) + 2(1 + \alpha)u(k - \sigma\ell) = 0, \ k \in [0, \infty), \tag{16}$$

where τ and σ are odd and even positive integers respectively. Equation (16) satisfies the assumptions of Theorem 3.5, and therefore the equation (16) is oscillatory. In fact, $(-\alpha)^{\left[\frac{k}{\ell}\right]+1}$ is an oscillatory solution of (16).

Theorem 3.7. *The conclusion of Theorem 3.5 holds even if (14) is replaced by*

$$\sum_{s=0}^{\infty} \big(\frac{1}{\alpha}\big)^s (K + s\ell) q(K + s\ell) \sum_{r=0}^{\infty} \big(\frac{1}{\alpha}\big)^r q(k + r\ell) = \infty. \tag{17}$$

Proof. Since (14) implies that the equation (1) is oscillatory, it suffices to show that all conditions of (1) oscillate in the case that

$$\sum_{r=0}^{\infty} \big(\frac{1}{\alpha}\big)^r q(K + r\ell) < \infty. \tag{18}$$

Assume for the sake of contradiction, that (1) has an eventually positive solution $u(k)$. Then, by Lemma (3.4)(a), $z(k) = u(k) - u(k - \tau\ell) > 0$ and $\Delta_{\alpha(\ell)} z(k) \leq 0$ eventually. Thus, eventually $u(k) > u(k - \tau\ell)$, which implies that there exists a constant $L > 0$ and $K \in [0, \infty)$ sufficiently large such that

$u(k-\ell\mu) \geq L$, $k \geq K$. Thus, from $\Delta_{\alpha(\ell)} z(k) = -q(k)u(k-\sigma\ell)$ it follows that $\Delta_{\alpha(\ell)} z(k) \leq -Lq(k)$, $k \geq K$ and hence $z(k) \geq L \sum_{r=0}^{\infty} \left(\frac{1}{\alpha}\right)^r q(k+r\ell)$, $k \geq K$, which is the same as

$$u(k) \geq \alpha u(k-\tau\ell) + L \sum_{r=0}^{\infty} \left(\frac{1}{\alpha}\right)^r q(k+r\ell), \ k \geq K. \tag{19}$$

Now let $I(k)$ denote the integer part of $\frac{k-K}{\tau}$, then we have

$$u(k) \geq L \Bigg(\sum_{r=0}^{\infty} \left(\frac{1}{\alpha}\right)^r q(k+r\ell) + \sum_{r=0}^{\infty} \left(\frac{1}{\alpha}\right)^r q(k+r\ell-\tau\ell) + \cdots$$

$$+ \sum_{r=0}^{\infty} \left(\frac{1}{\alpha}\right)^r q(k+r\ell-(I(k)-1)\tau\ell) \Bigg) + u(k-I(k)\tau\ell),$$

which together with $\Delta_{\alpha(\ell)} z(k) = -q(k)u(k-\sigma\ell)$ yields

$$\Delta_{\alpha(\ell)} z(k) \leq H(k), \tag{20}$$

$$\text{where} \quad H(k) = I(k)Lq(k) \sum_{r=0}^{\infty} \left(\frac{1}{\alpha}\right)^r q(k+r\ell).$$

By noting the fact that $I(k)/k \to 1/\tau\ell$ as $k \to \infty$, we have

$$H(k) \Big(kq(k) \sum_{r=0}^{\infty} \left(\frac{1}{\alpha}\right)^r q(k+r\ell) \Big)^{-1} = \frac{I(k)L}{k} \to \frac{L}{\tau\ell} \text{ as } k \to \infty. \tag{21}$$

Thus (17) and (21) imply that $\sum_{r=0}^{\infty} \left(\frac{1}{\alpha}\right)^r H(K+r\ell) = \infty$, which together with (20) leads to $z(k) \to -\infty$ as $k \to \infty$. This contradicts the hypothesis that $z(k)$ is eventually positive. □

Example 3.8. For the generalized neutral α-difference equation

$$\Delta_{\alpha(\ell)}(u(k) - \alpha u(k-\tau\ell)) + k^{-\eta\ell} u(k-\sigma\ell) = 0, \quad \eta \in (1, 3/2] \tag{22}$$

condition (17) is satisfied. Therefore, by Theorem 3.7 the equation (22) is oscillatory. However, the condition (14) does not satisfy.

Acknowledgements

For the first author, research is supported by National Board for Higher Mathematics, Department of Atomic Energy, Mumbai, India.

References

[1] R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, 2000.

[2] J. Popenda and B. Szmanda, On the oscillation of solutions of certain difference equations, *Demonstratio Mathematica*, **XVII**, No 1 (1984), 153-164.

[3] M. Maria Susai Manuel, G. Britto Antony Xavier and E. Thandapani, Theory of generalized differenceoperator and its applications, *Far East Journal of Mathematical Sciences*, **20**, No 2 (2006), 163-171.

[4] M. Maria Susai Manuel, G. Britto Antony Xavier and E. Thandapani, Qualitative properties of solutions of certain class of difference equations, *Far East Journal of Mathematical Sciences*, **23**, No 3 (2006), 295-304.

[5] M. Maria Susai Manuel, G. Britto Antony Xavier and E. Thandapani, Generalized Bernoulli polynomials through weighted Pochhammer symbols, *Far East Journal of Applied Mathematics*, **26**, No 3 (2007), 321-333.

[6] M. Maria Susai Manuel, A. George Maria Selvam and G. Britto Antony Xavier, Rotatory and boundedness of solutions of certain class of difference equations, *International Journal of Pure and Applied Mathematics*, **33**, No 3 (2006), 333-343.

[7] M. Maria Susai Manuel and G. Britto Antony Xavier, Recessive, dominant and spiral behaviours of solutions of certain class of generalized difference equations, *International Journal of Differential Equations and Applications*, **10**, No 4 (2007), 423-433.

[8] M. Maria Susai Manuel, V. Chandrasekar and G. Britto Antony Xavier, Solutions and applications of certain class of α-difference equations, *International Journal of Applied Mathematics*, **24**, No 6 (2011), 943-954.

[9] Ronald E. Mickens, *Difference Equations*, Van Nostrand Reinhold Company, New York, 1990.