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CONSTRUCTION OF EXTENDED EXPONENTIAL GENERAL LINEAR METHODS

U.A. Osisiogu[1], F.E. Bazuaye[2] §

[1]Department of Industrial Mathematics and Applied Statistics
Ebonyi State University, Abakaliki
Ebonyi State, NIGERIA

[2]Department of Mathematics and Statistics
University of Portharcourt, Portharcourt
Rivers State, NIGERIA

Abstract: This paper is concerned with the construction and numerical analysis of Extended Exponential General Linear Methods (EEGLM). These methods are related to the methods of Butcher [1] and Calvo and Palencia [3] but, in contrast to the latter, we make use of higher terms of the exponential and related matrix functions. This feature enables us to derive the order conditions which in turn aided in the construction of family of methods of higher order. The numerical experiments indicate that Extended Exponential General Linear Methods perform better than the existing methods.

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1. Introduction

Although the theory of numerical methods for time integration is well established for a general class of problems, recently due to improvements in the

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§Correspondence author

efficient computation of the exponential function, exponential integrators for problems

$$y'(t) = Ly(t)) + N(y(t)), \quad 0 \leq t \leq T, \text{ given } y(0),$$

have emerged as a viable alternative. In the early fifties, the phenomenon of stiffness was first discovered by Curtis and Hirschfelder [4]. The stiffiness effectively yields explicit integrators useless, as stability rather than accuracy governs how the integrator performs. It could be said that more integrators have been developed to overcome the phenomenon of stiffiness, than any other property that a differential equation may have. Butcher and Wright [2], provided a novel approach to solving stiff problems. Integrators were constructed, which solve exactly the linear part of the problem and then used a change of variables to cast the problem in a form, which a traditional explicit method can be used to solve the transformed equation, the approximate solution is then back transformed. These methods are commonly known as Integrating Factor (IF) methods. Calvo and Palencia [3] constructed a class of methods known as the RKMK methods. Enright and Muir [5] introduced the Generalized Integrating Factor (GIF) methods which were shown to exhibit large improvements in accuracy over the, IF, ETD and CF methods. It was concluded that the ETD methods constructed though stable but could be improved on, so as to be able to construct higher order.

2. Mathematical Formulation

In the past few years the exponential General Linear Methods for the problem

$$y'(t) = Ly(t)) + N(y(t)), \quad 0 \leq t \leq T, \text{ given } \; y(0), \tag{1}$$

have attracted a lot of interest. Hence, this paper dwells on the construction of extended exponential general linear methods (EEGLM) for the autonomous problem (1).

For given starting values $y_0, y_1, \cdots y_{q-1}$, the theoretical approximation y_{n+1} at time t_{n+1},$n \leq q-1$, is given by the recurrence relation or formula

$$y_{n+1} = e^{hl} y_n + h \sum_{i=1}^{s} B_i(hL) N(Y_{ni}) + h \sum_{k=1}^{q-1} V_k(hL) N(y_{n-k}) \; . \tag{2}$$

The internal stages Y_{ni}, $1 \leq i \leq s$, are defined through

$$Y_{ni} = e^{c_i hl} y_n + h \sum_{j=1}^{i-1} A_{ij}^{(1)}(hl) N(y_{nj}) + h \sum_{k=1}^{q-1} U_{ik}^{(1)}(hl) N(y_{n-k})$$

$$+h^2\sum_{j=1}^{i-1} A_{ij}^{(2)}(hl)N'(y_{nj}) + h^2\sum_{k=1}^{q-1} U_{ik}^{(2)}(hl)N'(y_{n-k}). \tag{3}$$

Throughout this paper we assume that these conditions are satisfied. The coefficients can be represented in the following table.

$$\begin{array}{c|c|c|c}
A_{21}^{(1)} & U_{21}^{(1)}\cdots U_{2,q-1}^{(1)} & A_{21}^{(2)} & U_{21}^{(2)}\cdots U_{2,q-1}^{(2)} \\
\vdots\ddots & \vdots\cdots\vdots & \vdots\ddots & \vdots\cdots\vdots \\
A_{s1}^{(1)}\cdots A_{s,s-1}^{(1)} & U_{s1}^{(1)}\cdots U_{s,q-1}^{(1)} & A_{s,s-1}^{(2)}\cdots A_{s,s-1}^{(2)} & U_{s1}^{(2)}\cdots U_{s,q-1}^{(2)} \\
\hline
B_1\cdots B_{s-1}B_s & V_1\cdots V_{q-1} & &
\end{array}$$

Table 1: Coefficients table

3. Order Conditions for the Extended Exponential General Linear Methods

Deriving the order conditions for the method (2), we assume the data to be sufficiently regular. In particular, we require that the nonlinearity evaluated at the exact solution $f(t) = N(y(t))$ is sufficiently differentiable with respect to t for $0 < t < T$. Before deriving the order conditions for this method, we state and prove the lemma below which will form the basis of proving the order conditions.

Lemma 1. *The exact solution of the initial value problem*

$$y' = Ly + f(t), t \geq t_n, \quad \textit{with given} \quad y(t_n), \tag{4}$$

has the following representation

$$y(t_n+\tau) = e^{l\tau}y(t_n) + \sum_{\ell=0}^{m-1}\tau^{(l+1)}\psi_{\ell+1}(l\tau)f^{(l)}(t_n) + R_m(m,\tau) \tag{5}$$

with

$$R_m(m,\tau) = \int_0^\tau r^{(\tau-\sigma)}\int_0^\sigma \frac{(\sigma-\xi)^{m-1}}{(m-1)!}f^{(m)}(t_m+\xi)d\xi d\sigma.$$

Proof.

$$f(t_n+\sigma)=t_n+\sigma f(t_n)+\frac{\sigma^2}{2!}f^{(2)}(t_n)+\cdots+\frac{\sigma^{m-1}}{(m-1)!}f^{(m-1)}(t_n)$$

$$+\int_0^{\sigma}\frac{(\sigma-\xi)^m}{(m!)}f^{(m)}(t_n+\xi)d\xi.$$

But

$$y(t_n+\tau)=e^{\tau L}y(t_n)+\int_0^{\tau}e^{(\tau-\sigma)}f(t_n+\sigma)d\sigma,\quad \tau\geq 0. \tag{6}$$

Inserting equation (6) into equation (7) above, we have

$$y(t_n+\tau)=e^{\tau L}y(t_n)+\int_0^{\tau}e^{(\tau-\sigma)}[f(t_n)+\sigma f(t_n)\frac{\sigma^2}{2!}f^{(2)}f(t_n)+\cdots$$

$$+\frac{\sigma^{m-1}}{(m-1)!}f^{m-1}f(t_n]d\sigma+\int_0^{\tau}\frac{(\sigma-\xi)^{m-1}}{(m-1)!}f^m f(t_n+\epsilon)d\sigma. \tag{7}$$

Applying the definition of the ψ functions, we have

$$y(t_n+\tau)=e^{\tau L}y(t_n)+\sum_{\ell=0}^{m-1}\tau^{\ell+1}\psi_{\ell+1}f^{(1)}(t_n)+R_n(m,\tau), \tag{8}$$

where

$$R_n(m,\tau)=\int_0^{\tau}r^{(\tau-\sigma)}\int_0^{\tau}\frac{(\sigma-\xi)^{m-1}}{(m-1)!}f^{(m)}f(t_n+\xi)d\sigma,\quad \tau\geq 0,$$

$$y_{n+1}=e^{hl}y_n+h\sum_{i=1}^{s}B_i(hL)N(Y_{ni})+h\sum_{k=1}^{q-1}V_k(hL)f(t_n-kh), \tag{9}$$

and the internal stages are defined through

$$Y_{ni}=e^{c_ihl}y_n+h\sum_{j=1}^{i-1}A_{ij}^{(1)}(hl)f(t_{nj})+h\sum_{k=1}^{q-1}U_{ik}^{(1)}(hl)f(t_n-kh)$$

$$+h^2\sum_{j=1}^{i-1}A_{ij}^{(2)}(hl)f''(t_n+hc_j)+h^2\sum_{k=1}^{q-1}U_{ik}^{(2)}(hl)f'(t_n-kh),\quad 1\leq i\leq S. \tag{10}$$

Expanding the functions in the equation above

$$f(t_n + c_j h) = f(t_n) + c_j h f'(t_n + \frac{(c_j h)^2}{2!} f''(t_n) + \frac{(c_j h)^3}{3!} f^{(3)}(t_n)$$

$$+ \cdots + \frac{(c_j h)^{m-1}}{(m-1)!} f^{(m-1)}(t_n) + R(s, m), \tag{11}$$

where

$$R(s, m) = \int_0^\tau \frac{(\sigma - \xi)^m}{m!} f^{(m)}(t_n + \xi) d\xi,$$

$$f(t_n - kh) = f(t_n) - (kh) f'(t_n + \frac{(-kh)^2}{2!} f''(t_n) + \frac{(-kh)^3}{3!} f^{(3)}(t_n)$$

$$+ \cdots + \frac{(-kh)^{m-1}}{(m-1)!} f^{(m-1)}(t_n) + R(s, r), \tag{12}$$

$$f'(t_n + c_j h) = f'(t_n) + c_j h f''(t_n) + \frac{(c_j h)^2}{2!} f^{(3)}(t_n) + \frac{(c_j h)^3}{3!} f^{(iv)}(t_n) + \cdots$$

$$+ \frac{(c_j h)^{m-1}}{(m-1)!} f^{(m)}(t_n) + R(s, m), \tag{13}$$

$$f'(t_n - kh) = f' t_n) + (-kh) f''(t_n + \frac{(-kh)^2}{2!} f^{(3)}(t_n) + \frac{(-kh)^3}{3!} f^{(iv)}(t_n)$$

$$+ \cdots + \frac{(-kh)^{m-1}}{(m-1)!} f^{(m)}(t_n) + R(s, m), \tag{14}$$

$$f(t_n + c_i h) = f(t_n) + (c_i h f'(t_n) + \frac{(c_i h)^2}{2!} f''(t_n) + \frac{(c_i h)^3}{3!} f^{(3)}(t_n)$$

$$+ \cdots + \frac{(c_i h)^{m-1}}{(m-1)!} f^{(m-1)}(t_n) + R(s, m). \tag{15}$$

Substituting where appropriate into (10) we have

$$y_{n+1} = e^{hL} y_n + h \sum_{i=1}^{s} B_i N(Y_{ni}) + h \sum_{k=1}^{q-1} V_k [f(t_n) - (kh) f'(t_n) + \frac{(-kh)^2}{2!} f''(t_n)$$

$$+\frac{(-kh)^3}{3!}f^{(3)}(t_n)+\cdots+\frac{(-kh)^{m-1}}{(m-1)!}f^{(m-1)}(t_n+R(s,r)], \tag{16}$$

$$Y_{ni}=e^{c_ihL}y_n+h\sum_{j=1}^{i-1}A_{ij}^{(1)}[f(t_n)+c_jhf'(t_n+\frac{(c_jh)^2}{2!}f''(t_n)+\frac{(c_jh)^3}{3!}f^{(3)}(t_n)$$

$$+\cdots+\frac{(c_jh)^{m-1}}{(m-1)!}f^{(m-1)}(t_n)+R(s,m)]$$

$$+h\sum_{k=1}^{q-1}U_{ik}^{(1)}[f(t_n)-(kh)f'(t_n+\frac{(-kh)^2}{2!}f''(t_n)+\frac{(-kh)^3}{3!}f^{(3)}(t_n)+$$

$$\cdots+\frac{(-kh)^{m-1}}{(m-1)!}f^{(m-1)}(t_n)+R(s,r)+h^2\sum_{j=1}^{i-1}A_{ij}^{(2)}[f'(t_n)+(-kh)f''(t_n)$$

$$+\frac{(-kh)^2}{2!}f^{(3)}(t_n)+\frac{(-kh)^3}{3!}f^{(iv)}(t_n)+\cdots+\frac{(-kh)^{m-1}}{(m-1)!}f^{(m)}(t_n)+R(s,m)]$$

$$+h^2\sum_{k=1}^{q-1}U_{ik}^{(2)}[f't_n)+(-kh)f''(t_n+\frac{(-kh)^2}{2!}f^{(3)}(t_n)+\frac{(-kh)^3}{3!}f^{(iv)}(t_n)$$

$$+\cdots+\frac{(-kh)^{m-1}}{(m-1)!}f^{(m)}(t_n)+R(s,m)]. \tag{17}$$

Substituting the exact solution values

$$y_n=y_n, Y_{ni}=y(t_n+c_ih), \quad 1\le i\le s, \quad n\ge 0, \tag{18}$$

$$y(t_n+c_ih)=e^{c_ihL}y_n+c_ih\int_0^1 e^{(1-\tau)c_ihL}f(t_n+c_ih)d\tau, \tag{19}$$

$$f(t_n+c_ih)=f(t_n)+(hc_i)f'(t_n)+\frac{(hc_i)^2}{2!}f''(t_n)+\frac{(hc_i)^3}{3!}f'''(t_n)$$

$$+\cdots+\frac{(hc_i)^{m-1}}{(m-1)!}f^{(m-1)}(t_n)+R(s,m). \tag{20}$$

Inserting (24) into (23), we get

$$y(t_n+c_ih)=e^{c_ihL}y_n+h\int_0^1 e^{(1-\tau)c_ihL}[f(t_n)+(hc_i)f'(t_n)+\frac{(hc_i)^2}{2!}f''(t_n)$$

$$+\frac{(hc_i)^3}{3!}f'''(t_n)+\cdots+\frac{(hc_i)^{m-1}}{(m-1)!}f^{(m-1)}(t_n)+R(s,m)]. \tag{21}$$

Similarly,

$$y'(t_n+c_ih)=e^{c_ihL}y_n+h\int_0^1 e^{(1-\tau)c_ihL}f'(t_n+c_ih), \tag{22}$$

$$f'(t_n+c_ih)=f'(t_n)+c_ihf''(t_n)+\frac{(c_ih)^2}{2!}f'''(t_n)$$

$$+\frac{(c_ih)^3}{3!}f^{(iv)}(t_n)+\cdots+\frac{(c_ih)^{m-2}}{(m-2)!}f^{(m-1)}(t_n)+R(s,m), \tag{23}$$

and inserting (24) into (23) we get

$$y'(t_n+c_ih)=e^{c_ihL}y_n+h\int_0^1 e^{(1-\tau)c_ihL}[f'(t_n)+c_ihf''(t_n)$$

$$+\frac{(c_ih)^2}{2!}f'''(t_n)+\frac{(c_ih)^3}{3!}f^{(iv)}(t_n)+\cdots+\frac{(c_ih)^{m-2}}{(m-2)!}f^{(m-1)}(t_n)+R(s,m)]. \tag{24}$$

Subtracting (18) from (20) which denotes the defect (Dni) of the stages and expressing them in terms of h, we have

$$h^1:=c_i\int_0^1 e^{(1-\tau)c_ihL}f(t_n)d\tau-\sum_{j=1}^{i-1}A_{ij}^{(1)}f(t_n)-\sum_{k=1}^{q-1}U_{ik}^{(1)}f(t_n),$$

$$h^2:=\frac{c_i^2}{2!}\int_0^1 e^{(1-\tau)c_ihL}f'(t_n)d\tau-c_j\sum_{j=1}^{i-1}A_{ij}^{(1)}f'(t_n)-(-k)\sum_{k=1}^{q-1}U_{ik}^{(1)}f'(t_n)$$

$$-\sum_{j=1}^{i-1}A_{ij}^{(2)}f'(t_n)-\sum_{k=1}^{q-1}U_{ik}^{(2)}f'(t_n),$$

$$h^3:=\frac{c_i^3}{3!}\int_0^1 e^{(1-\tau)c_ihL}f''(t_n)d\tau-\frac{c_j^2}{2!}\sum_{j=1}^{i-1}A_{ij}^{(1)}f''(t_n)-\frac{(-k)^2}{2!}$$

$$\times\sum_{k=1}^{q-1}U_{ik}^{(1)}f''(t_n)-c_j\sum_{j=1}^{i-1}A_{ij}^{(2)}f''(t_n)-(-k)\sum_{k=1}^{q-1}U_{ik}^{(2)}f''(t_n),$$

$$\vdots$$

$$h^m := \frac{c_i^m}{m!}\int_0^1 e^{(1-\tau)c_ihL} f^{(m-1)}(t_n)d\tau - \frac{c_j^{m-1}}{(m-1)!}\sum_{j=1}^{i-1} A_{ij}^{(1)} f^{(m-1)}(t_n) - \frac{(-k)^{m-1}}{(m-1)!}$$

$$\times \sum_{k=1}^{q-1} U_{ik}^{(1)} f^{(m-1)}(t_n) - \frac{c_j^{m-2}}{(m-2)!}\sum_{j=1}^{i-1} A_{ij}^{(2)} f^{(m-1)}(t_n)$$

$$- \frac{(-k)^{m-2}}{(m-2)!}\sum_{k=1}^{q-1} U_{ik}^{(2)} f^{(m-1)}(t_n), \tag{25}$$

$$D_{ni} = hc_i^1\psi_1(c_ihL) + h^2c_i^2\psi_2(c_ihL) + h^3c_i^3\psi_3(c_ihL) + \cdots + h^Qc_i^Q\psi_Q(c_ihL)$$

$$-\sum_{j=1}^{i-1}\frac{c_j^{\ell-1}}{(\ell-1)!}A_{ij}^{(1)}(hL)f^{(\ell-1)}(t_n)$$

$$-\sum_{k=1}^{q-1}\frac{(-k)^{\ell-1}}{(\ell-1)!}U_{ik}^{(1)}(hL)f^{(\ell-1)}(t_n) - \sum_{j=1}^{i-1}\frac{c_j^{\ell-2}}{(\ell-2)!}A_{ij}^{(2)}(hL)f^{(\ell-1)}(t_n) \tag{26}$$

$$-\sum_{k=1}^{q-1}\frac{(-k)^{\ell-2}}{(\ell-2)!}U_{ik}^{(2)}(hL)f^{(\ell-1)}(t_n),$$

$$D_{ni} = \sum_{\ell-1}^{Q} h^\ell M_{\ell i}(hL)f^(\ell-1)(t_n), \tag{27}$$

where

$$M_{\ell i}(hL) = c_i^\ell\psi_\ell(c_ihL) - \sum_{j=1}^{i-1}\frac{c_j^{\ell-1}}{(\ell-1)!}A_{ij}^{(1)} - [\sum_{k=1}^{q-1}\frac{(-k)^{\ell-1}}{(\ell-1)!}U_{ik}^{(1)}(hL)]$$

$$-\sum_{j=1}^{i-1}\frac{c_j^{\ell-1}}{(\ell-1)!}A_{ij}^{(2)} - \sum_{k=1}^{q-1}\frac{(-k)^{\ell-1}}{(\ell-1)!}U_{ik}^{(2)}(hL). \tag{28}$$

Likewise, the numerical solution defects equals

$$d_{n+1} = \sum_{\ell=1}^{p} h^\ell v_\ell(hL)f^{(\ell-1)}(t_n) + \cdots, \tag{29}$$

where

$$v_\ell(hL) = \psi_\ell(hL) - \sum_{\ell=1}^{s} \frac{c_i^{\ell-1}}{(\ell-1)!} B_i(hL) - \sum_{k=1}^{q-1} \frac{(k-1)^{\ell-1}}{(\ell-1)!} V_k(hL). \tag{30}$$

$\square$

Definition 2. Our numerical scheme (2) is said to be of stage order Q and order P if $D_{ni} = 0(h^{Q+1})$ and $d_{n+1} = 0(h^{P+1})$. That is, requiring $M_{\ell i}(hL) = 0$ and $v_\ell(hL) = 0$, we obtain the order conditions

$$c_i^\ell \psi_\ell(c_i hL) = \sum_{j=1}^{i-1} \frac{c_j^{\ell-1}}{(\ell-1)!} A_{ij}^{(1)}(hL) + \sum_{k=1}^{q-1} \frac{(-k)^{\ell-1}}{(\ell-1)!} U_{ik}^{(1)}(hL)$$

$$+ \sum_{j=1}^{i-1} \frac{c_j^{\ell-2}}{(\ell-2)!} A_{ij}^{(2)}(hL) + \sum_{k=1}^{q-1} \frac{(-k)^{\ell-2}}{(\ell-2)!} U_{ik}^{(2)}(hL), \tag{31}$$

$$\psi_\ell(hL) = \sum_{i=1}^{s} \frac{c_j^{\ell-1}}{(\ell-1)!} B_i(hL) + \sum_{k=1}^{q-1} \frac{(-k)^{\ell-1}}{(\ell-1)!} V_k(hL), \tag{32}$$

and so by definition $c_i = 0$ for all $1 \leq i \leq s$.

3.1. Construction of Extended Exponential General Linear Methods of Order Three Step Two Stage Order Two

The extended exponential general linear methods order three step two stage order two (known as methods 322) are given as

$$y_{n+1} = e^{hL} y_n + hB_1(hL)N(Y_{n1}) + hB_2(hL)N(Y_{n2}) + hV_1 N(y_{n-1}),$$

$$Y_{n2} = e^{c_2 hL} y_n + hA_{21}(hL)N(Y_{n1}) + hU_{21}^{(1)}(hL)N(y_{n-1})$$

$$+ h^2 A_{21}^{(2)}(hL)N'(y_{n1}) + h^2 U_{21}^{(2)}(hL)N'(y_{n-1}). \tag{33}$$

Again, making use of the order conditions (32) and (33), we have

$$c_1^1 A_{21}^{(1)} + (-1)^1 U_{21}^{(1)} + c_1^0 A_{21}^{(2)} + (-1)^0 U_{21}^{(2)} = \psi_2,$$

$$-U_{21}^{(1)} + A_{21}^{(2)} + U_{21}^{(2)} = \psi_2, \tag{34}$$

$$\frac{c_1^2}{2!}A_{21}^{(1)} + \frac{(-1)^2}{2!}U_{21}^{(1)} + c_1^1 A_{21}^{(2)} + (-1)U_{21}^{(2)} = c_2^2\psi_3(c_2hL),$$

$$\frac{1}{2}U_{21}^{(1)} - U_{21}^{(2)} = \psi_3, \tag{35}$$

$$\frac{c_1^3}{3!}A_{21}^{(1)} + \frac{c_1^2}{2!}A_{21}^{(2)} + \frac{(-1)^3}{3!}U_{21}^{(1)} + \frac{(-1)^2}{2!}U_{21}^{(2)} = c_2^3\psi_4(c_2hL),$$

$$\frac{1}{6}U_{21}^{(1)} + \frac{1}{2}U_{21}^{(2)} = \psi_4. \tag{36}$$

Similarly,

$$c_1^0 B_1 + c_2^0 B_2 + (-1)^0 V_1 = \psi_1$$

$$B_1 + B_2 + V_1 = \psi_1 \tag{37}$$

$$c_1^1 B_1 + c_2^1 B_2 + (-1)^1 V_1 = \psi_2$$

$$B_2 - V_1 = \psi_2 \tag{38}$$

$$\frac{1}{2!}c_1^2 B_1 + \frac{1}{2!}c_2^0 B_2 + \frac{1}{2!}(-1)^0 V_1 = \psi_3 \tag{39}$$

$$\frac{1}{2}B_2 + \frac{1}{2}V_1 = \psi_3. \tag{40}$$

This gives

$$A_{21}^{(2)} = 5\psi_2 + 6\psi_3$$
$$U_{21}^{(1)} = 5(\psi_2 + 2\psi_3)$$
$$U_{21}^{(2)} = 2(\psi_2 + 3\psi_3)$$
$$B_1 = \psi_1 - 2\psi_3$$
$$B_2 = \frac{1}{2}(\psi_2 + \psi_3)$$
$$V_1 = \frac{1}{2}(-\psi_1 + 2\psi_2).$$

This is represented in the following table.

0	$5(\psi_2+2\psi_3)$	$(5\psi_2+6\psi_3)$	$2(\psi_2+3\psi_3)$
$(\psi_1-2\psi_3)\frac{1}{2}(\psi_2+\psi_3)$	$\frac{1}{2}(-\psi_1+2\psi_2)$		

Table 2: The coefficients of the extended Exponential General Linear methods order three step two and stage order two (322)

3.2. Construction of Extended Exponential General Linear Methods Order Five Step Two Stage Order Three

The extended exponential general linear methods order five step two stage order three (known as method 523) are given as

$$y_{n+1} = e^{hL}y_n + hB_1(hL)N(y_n) + hV_1N(y_{n-1}) + hV_2N(y_{n-2}), \tag{40}$$

$$\begin{aligned} Y_{n2} = e^{c_2hL}y_n + hA_{21}N(y_n) + hU_{21}^{(1)}N(y_{n-1}) + hU_{22}^{(1)}N(y_{n-2}) \\ +h^2U_{21}^{(2)}N'(y_{n-1}) + h^2U_{22}^{(2)}N'(y_{n-2}), \end{aligned} \tag{41}$$

using the order conditions (32) and (33) to determine the coefficient matrix

$$\begin{aligned} c_1^1A_{21}^{(1)} + (-1)^1U_{21}^{(1)} + (-2)^1U_{22}^{(1)} + c_1^0A_{21}^{(2)} \\ +(-1)^0U_{21}^{(2)} + (-2)^0U_{22}^{(2)} = \psi_2, \end{aligned}$$

$$\text{therefore,} \quad -U_{21}^{(1)} - 2U_{22}^1 + A_{21}^{(2)} + U_{21}^{(2)}U_{22}^{(2)} = \psi_2, \tag{42}$$

$$c_1^2A_{21}^{(1)} + \frac{(-1)^1}{2!}U_{21}^{(1)} + \frac{(-2)^2}{2!}U_{22}^{(1)} + (-2)U_{22}^{(2)} = c_2^3\psi_3(c_2hL),$$

$$\frac{1}{2}U_{21}^{(1)} + 2U_{22}^{(1)} - U_{21}^{(2)} - 2U_{22}^{(2)} = \psi_3, \tag{43}$$

$$c_1^3A_{21}^{(1)} + \frac{(-1)^3}{3!}U_{21}^{(1)} + \frac{(-2)^3}{3!}U_{22}^{(1)} + \frac{(-2)^2}{2!}U_{21}^{(2)} + \frac{(-2)^2}{2!}U_{22}^{(1)} = c_2^4\psi_4(c_2hL),$$

$$\frac{-1}{6}U_{21}^{(1)} + \frac{-8}{6}U_{22}^{(1)} + \frac{1}{2}U_{21}^{(2)} + 2U_{22}^{(2)} = \psi_4, \tag{44}$$

$$c_1^4A_{21}^{(1)} + \frac{(-1)^4}{4!}U_{21}^{(1)} + \frac{(-2)^4}{4!}U_{22}^{(1)} + \frac{(-1)^3}{3!}U_{21}^{(2)} + \frac{(-2)^3}{3!}U_{22}^{(2)} = c_2^5\psi_5(c_2hL),$$

$$\frac{-1}{24}U_{21}^{(1)} + \frac{16}{24}U_{22}^{(1)} + \frac{-1}{6}U_{21}^{(2)} + \frac{-8}{6}U_{22}^{(2)} = \psi_5. \tag{45}$$

Solving the equations above equations gives

$$B_1 = (\psi_1 + \frac{1}{6}\psi_2 - 3\psi_3 - \psi_4),$$

$$B_2 = \frac{1}{2}(\psi_2 + 2\psi_3),$$

$$V_1 = \frac{1}{6}(-5\psi_2 + 6\psi_3 + 12\psi_4),$$

$$V_2 = \frac{1}{6}(\psi_2 - \psi_4).$$

3.3. Construction of Extended Exponential General Linear Methods Order Five Step Two Stage Order Four

The extended exponential general linear methods order five step two stage order four (known as methods 524) are given as

$$y_{n+1} = e^{hL}y_n + hB_1(hL)N(y_n) + hB_2(hL)N(Y_{n2}) + hV_1N(y_{n-1})$$

$$= hV_2N(y_{n-2}) + hV_3N(y_{n-3}), \tag{46}$$

$$Y_{n2} = e^{c_2hL}y_n + hA_{21}N(y_n) + hU_{21}^{(1)}N(y_{n-1}) + hU_{22}^{(1)}N(y_{n-2}) + hU_{23}^{(1)}N(y_{n-3})$$

$$+h^2A_{21}^{(2)}N'(y_n) + h^2U_{21}^{(2)}N'(y_{n-1}) + h^2U_{22}^{(2)}N'(y_{n-2}) + h^2U_{23}^{(2)}N'(y_{n-3}), \tag{47}$$

using the order conditions (32) and (33) to determine the coefficient matrix, we have

$$c_1^1A_{21}^{(1)} + (-1)^1U_{21}^{(1)} + (-2)^1U_{22}^{(1)} + (-3)^1U_{23}^{(1)} + c_1^0A_{21}^{(2)}$$
$$+(-1)^0U_{21}^{(2)}(-2)^0U_{22}^{(2)} + (-3)^0U_{23}^{(2)} = \psi_2,$$

$$-U_{21}^{(1)} - 2U_{22}^1 + A_{21}^{(2)} + U_{21}^{(2)} + U_{22}^{(2)} + U_{23}^{(2)} = \psi_2, \tag{48}$$

$$c_1^2\frac{1}{2!}A_{21}^{(1)} + \frac{(-1)^2}{2!}U_{21}^{(1)} + \frac{(-2)^2}{2!}U_{22}^{(1)} + \frac{(-3)^2}{2!}U_{23}^{(1)}$$

$$+(-1)^1U_{21}^{(2)} + (-2)^{(1)}U_{22}^{(2)} + (-3)^1U_{23}^{(2)} = \psi_3,$$

$$\frac{1}{2}U_{21}^{(1)} + 2U_{22}^{(1)} + \frac{9}{2}U_{23}^{(1)} - U_{21}^{(1)} - 2U_{22}^{(2)} - 3U_{23}^{(2)} = \psi_3, \tag{49}$$

$$\frac{1}{3!}c_1^3A_{21}^{(1)} + \frac{(-1)^3}{3!}U_{21}^{(1)} + \frac{(-2)^3}{3!}U_{22}^{(1)} + \frac{(-3)^3}{3!}U_{23}^{(1)}$$
$$+\frac{(-1)^2}{2!}U_{21}^{(2)} + \frac{(-2)^2}{3!}U_{22}^{(2)} + \frac{(-3)^2}{3!}U_{23}^{(2)} = \psi_4,$$

therefore $$\frac{-1}{3!}U_{21}^{(1)} + \frac{-8}{3!}U_{22}^{(1)} + \frac{-27}{3!}U_{23}^{(1)} + \frac{1}{2!}U_{21}^{(2)} + 2U_{22}^{(2)} + \frac{9}{2}U_{23}^{(2)} = \psi_4, \tag{50}$$

$$\frac{1}{4!}c_1^4A_{21}^{(1)}\frac{(-1)^4}{4!}U_{21}^{(1)} + \frac{(-2)^4}{4!}U_{22}^{(1)} + \frac{(-3)^4}{4!}U_{23}^{(1)},$$
$$\frac{1}{3!}c_1^3A_{21}^{(2)}\frac{(-1)^3}{3!}U_{21}^{(2)} + \frac{(-2)^3}{3!}U_{22}^{(2)} + \frac{(-3)^3}{3!}U_{23}^{(2)} = \psi_5,$$

therefore $$\frac{1}{4!}U_{21}^{(1)} + \frac{-16}{4!}U_{22}^{(1)} + \frac{-81}{4!}U_{23}^{(1)} + \frac{-1}{3!}U_{21}^{(2)} + \frac{-8}{3!}U_{22}^{(2)} + \frac{-27}{3!} = \psi_5, \tag{51}$$

$$\frac{1}{5!}c_1^5A_{21}^{(1)} + \frac{(-1)^5}{5!}U_{21}^{(1)} + \frac{(-2)^5}{5!}U_{22}^{(1)} + \frac{(-3)^5}{5!}U_{23}^{(1)}$$
$$+\frac{(-1)^4}{4!}U_{21}^{(2)} + \frac{(-2)^4}{4!}U_{22}^{(2)} + \frac{(-3)^4}{4!}U_{23}^{(2)} = \psi_6,$$

$$\frac{(-1}{)}5!U_{21}^{(1)} - \frac{32}{5!}U_{22}^{(1)} - \frac{243}{5!}U_{23}^{(1)} + \frac{1}{4!}U_{21}^{(2)} + \frac{16}{4!}U_{22}^{(2)} + \frac{81}{4!}U_{23}^{(2)} = \psi_6, \tag{52}$$

$$\frac{1}{6!}c_1^6A_{21}^{(1)} + \frac{(-1)^6}{6!}U_{21}^{(1)} + \frac{(-2)^6}{6!}U_{22}^{(1)} + \frac{(-3)^6}{6!}U_{23}^{(1)}$$
$$+\frac{(-1)^5}{5!}U_{21}^{(2)} + \frac{(-2)^5}{5!}U_{22}^{(2)} + \frac{(-3)^5}{5!}U_{23}^{(2)} = \psi_7,$$

$$\frac{1}{6!}U_{21}^{(1)} + \frac{64}{6!}U_{22}^{(1)} + \frac{729}{6!}U_{23}^{(1)} + \frac{1}{5!}U_{21}^{(2)} - \frac{32}{5!}U_{22}^{(2)} - \frac{243}{5!}U_{23}^{(2)} = \psi_7. \tag{53}$$

Similarly,

$$c_1^0B_1 + c_2^0B_2 + (-1)^0V_1 + (-2)^0V_2 + (-3)^0V_3 = \psi_1,$$

$$\text{therefore} \quad B_1 + B_2 + V_1 + V_2 + V_3 = \psi_1, \tag{54}$$

$$c_1^1 B_1 + c_2^1 B_2 + (-1)^1 V_1 + (-2)^1 V_2 + (-3)^1 V_3 = \psi_2,$$

$$\text{therefore} \quad B_2 - V_1 - 2V_2 - 3V_3 = \psi_2, \tag{55}$$

$$\frac{1}{2!}c_1^2 B_1 + \frac{1}{2!}c_2^2 B_2 + \frac{1}{2!}(-1)^2 V_1 + \frac{1}{2!}(-2)^2 V_2 + \frac{1}{2!}(-3)^2 V_3 = \psi_3,$$

$$\text{therefore} \quad \frac{1}{2}B_2 + \frac{1}{2}V_1 + 2V_2 + \frac{9}{2}V_3 = \psi_3, \tag{56}$$

$$\frac{1}{3!}c_1^3 B_1 + \frac{1}{3!}c_2^3 B_2 + \frac{1}{3!}(-1)^3 V_1 + \frac{1}{3!}(-2)^3 V_2 + \frac{1}{3!}(-3)^3 V_3 = \psi_4,$$

$$\text{therefore} \quad \frac{1}{6}B_2 - \frac{1}{6}V_1 - \frac{8}{6}V_2 - \frac{27}{6}V_3 = \psi_4, \tag{57}$$

$$\frac{1}{4!}c_1^4 B_1 + \frac{1}{4!}c_2^4 B_2 + \frac{1}{4!}(-1)^4 V_1 + \frac{1}{4!}(-2)^4 V_2 + \frac{1}{4!}(-3)^4 V_3 = \psi_5,$$

$$\frac{1}{24}B_2 + \frac{1}{24}V_1 + \frac{16}{24}V_2 + \frac{81}{24}V_3 = \psi_5. \tag{58}$$

Simplifying the above equations, we have

$$B_1 = \frac{1}{6}(6\psi_1 + 5\psi_2 - 10\psi_4 - 24\psi_5),$$

$$B_2 = \frac{1}{12}(3\psi_2 + 11\psi_3 + 18\psi_4 + 12\psi_5)$$

$$V_1 = \frac{1}{2}(-3\psi_2 + \psi_3 + 12\psi_4 + 12\psi_5)$$

$$V_2 = \frac{1}{6}(3\psi_2 + 2\psi_3 - 18\psi_4 - 25\psi_5)$$

$$V_3 = \frac{1}{12}(-\psi_2 - \psi_3 + 6\psi_4 + 12\psi_5).$$

Also,

$$U_{21}^{(1)} = \frac{3}{23}(6\psi_2 + 44\psi_3 + 157\psi_4 + 336\psi_5 + 420\psi_6 + 240\psi_7)$$

$$U_{22}^{(1)} = \frac{3}{46}(66\psi_2 + 277\psi_3 + 692\psi_4 + 936\psi_5 + 480\psi_6 - 120\psi_7)$$

$$U_{23}^{(1)} = \frac{1}{207}(78\psi_2 + 1124\psi_3 + 2961\psi_4 + 1608\psi_5 - 5580\psi_6 - 7920\psi_7)$$
$$U_{21}^{(2)} = \frac{6}{23}(18\psi_2 + 63\psi_3 + 126\psi_4 + 157\psi_5 + 110\psi_6 + 30\psi_7)$$
$$U_{22}^{(2)} = \frac{3}{23}(9\psi_2 - 3\psi_3 - 6\psi_4 + 136\psi_5 + 400\psi_6 + 360\psi_7)$$
$$U_{23}^{(2)} = \frac{3}{69}(6\psi_2 + 67\psi_3 + 180\psi_4 + 129\psi_5 - 270\psi_6 - 450\psi_7).$$

4. Numerical Experiments

In this section, we illustrate the accuracies of different extended exponential general linear methods.

Problem 1.
$$y' = \frac{y}{4}(1 - \frac{y}{20}), y(0) = 1$$
with explicit solution given as
$$y = \frac{20}{1 + 19\exp(-t^4)}.$$

Our scheme 322 exhibits remarkable improvement over Calvo and Palencia' results.

The graph of Extended Exponential General Linear Method 523 and Calvo and Palencia.

The graph of Extended Exponential General Linear Method 322, 523 and 524.

Problem 2. Given
$$y' = -10(y-1)^2, \quad y(0) = 2 \ \text{ with } \ t \in [0,1],$$
with exact solution
$$y = \frac{2+10t}{1+10t}.$$

The graph of EEGKM 524, CEERK and CRKM.

Our scheme 524 compares favorably with the convergent Explicit Runge-Kutta methods and Classical Runge-Kutta methods.

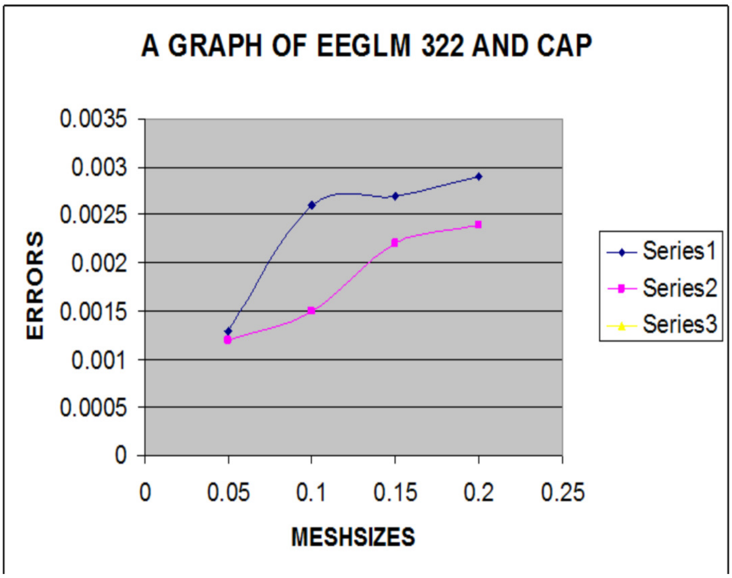

Figure 1

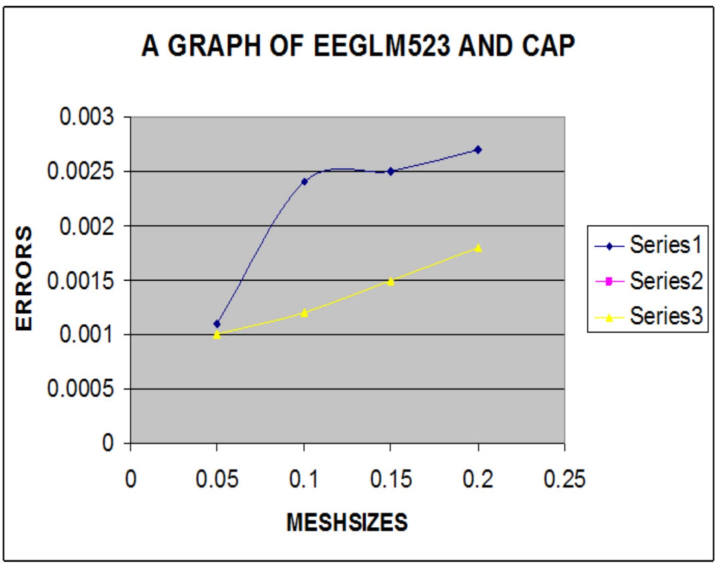

Figure 2

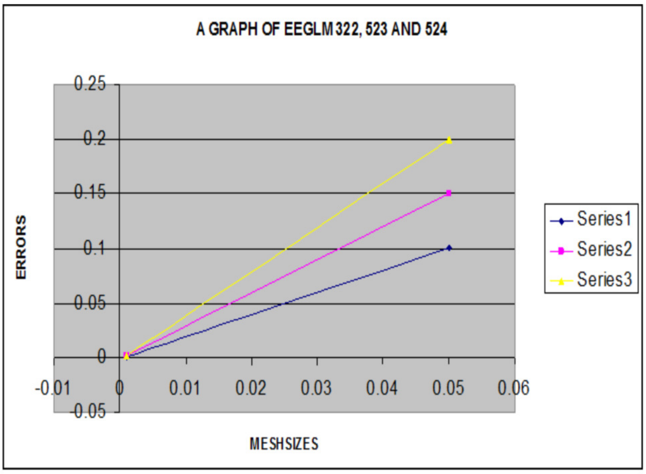

Figure 3

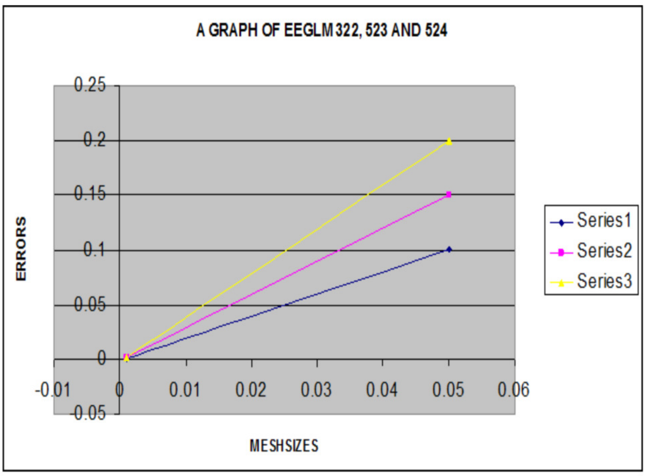

Figure 4

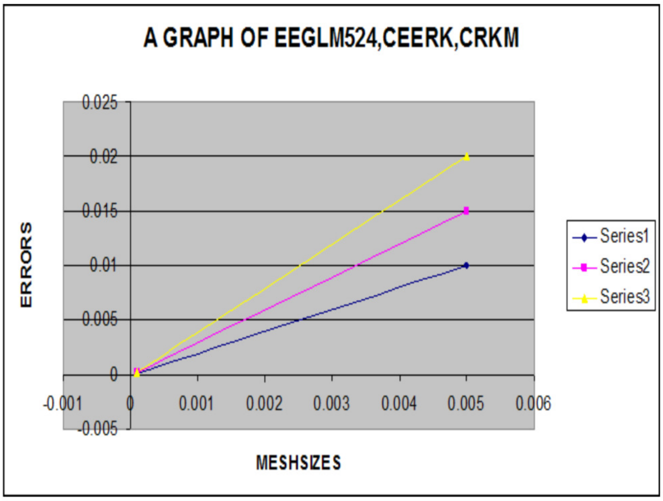

Figure 5

5. Conclusions

In this paper we have investigated the construction of Extended Exponential General linear methods using variation of constant formula. Its order conditions were derived. These order conditions form the basis for the construction of methods of higher order. The experimental experience reveals that our methods perform better than the already existing ones.

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