

A GENERALIZED VASICEK-MALKIEL BOND PRICING MODEL

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Abstract: Vasicek [14] was the first to propose a three constant-parameter one-factor short rate model for the evolution of interest rates and it was the first model to incorporate *mean reversion*, an essential characteristic of interest rates. We consider a generalization of Vasicek model in which the *normal level* of the short rate is an exponentially weighted average of past short rates as suggested in the work of Malkiel [12]. The differential equation giving the price of a zero-coupon bond by this *generalized Vasicek-Malkiel model* is derived. A complete explicit solution is then obtained as well as the yield curve. We then consider a further extension of the generalized Vasicek-Malkiel model by letting one of the parameters to be time dependent and utilize it to incorporate today's term structure of interest rates into the model and, again, an explicit solution is obtained.

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1. Introduction

Vasicek [14] was the first to propose a short rate model for the evolution of interest rates described by the stochastic differential equation:

$$dr = \gamma(\theta - r)dt + \sigma dX. \quad (1.1)$$

Here, σ is volatility of interest rate and dX is a Weiner process drawn from a

normal distribution with mean zero and variance dt . This is a three constant-parameter one-factor short rate model for the evolution of interest rates and it was the first model to incorporate *mean reversion*, an essential characteristic of interest rates. We consider a generalization of Vasicek model in which the *normal level* of the short rate is an exponentially weighted average of past short rates as suggested by Malkiel [12]; see also Cox, Ingersoll and Ross [3]. Hull and White [9] considered an extended Vasicek model (1.1) in which all the three parameters γ , θ and σ are time-dependent and the parameters are chosen to fit spot rate volatility, yield curve volatility, etc. Their model is a no-arbitrage model that is able to fit to today's term structure of interest rates. Hull and White [10] also proposed another variation of the model of the form:

$$dr = [\theta(t) - ar] dt + \sigma dX, \quad (1.2)$$

which fits the initial yield curve describing the current term structure of interest rates and updates the parameters as they step through time. For more discussion of interest rate models and their solutions see Black, Derman and Toy [1], Chawla [2], Duffie and Kan [4], Heath, Jarrow and Morton [5], Ho and Lee [6], Hughston [7], Hull [8], Klugman [11] and Mamon [13].

In the present paper we consider a generalized Vasicek model for the short term rate of interest, in which the normal level of the short rate is an exponentially weighted average of past short rates, described by the stochastic differential equation:

$$\begin{aligned} dr &= [\eta(t) + \gamma \{\theta(t) - r\}] dt + \sigma dX \\ &= u(r, \theta) dt + \sigma dX, \quad \text{say,} \end{aligned} \quad (1.3)$$

with γ and σ constant and where $\theta(t)$ is an exponentially weighted average of the past short rates:

$$\theta(t) = \mu \int_{-\infty}^t e^{-\mu(t-s)} r(s) ds, \quad (1.4)$$

with a parameter $\mu > 0$. In terms of a known $\theta(t_0)$, this can alternatively be written as

$$\theta(t) = e^{-\mu(t-t_0)} \theta(t_0) + \mu \int_{t_0}^t e^{-\mu(t-s)} r(s) ds, \quad (1.5)$$

and the differential equation form for (1.4) is

$$d\theta = \mu[r(t) - \theta(t)] dt = v(r, \theta) dt, \quad \text{say.} \quad (1.6)$$

The parameter $\eta(t)$ in (1.3) is introduced to fit the model to today's yield curve describing current term structure of interest rates. We call the interest rate model (1.3) a *generalized Vasicek-Malkiel model* for pricing of bonds.

In Section 2 the differential equation giving the price of a zero-coupon bond by this generalized Vasicek-Malkiel model is derived. A complete explicit solution is then obtained as well as the yield curve for the case of a constant η . In Section 3 we then consider a further extension of the generalized Vasicek-Malkiel model by letting the parameter η to be time dependent and utilize it to incorporate today's term structure of interest rates into the model and, again, an explicit solution is obtained.

2. Generalized Vasicek-Malkiel Model for Pricing of Zero-Coupon Bonds

Let $V(t, T; r, \theta)$, or $V(t, T)$ for short, denote the value of a zero-coupon bond at time t with maturity T , $t < T$, and final value $V(T, T) = Z$. In this section we consider η to be constant. A measure of future values of interest rates is the *yield curve*. Define

$$Y(t, T) = -\frac{1}{\tau} \ln \left(\frac{V(t, T)}{V(T, T)} \right). \quad (2.1)$$

Here, and in the following, we set time to maturity $\tau = T - t$. Whenever a function $F(t, T)$ is only a function of τ we write interchangeably $F(t, T) = F(\tau)$. Yield curve is the plot of Y against τ , and dependence of yield curve on τ is often called *term structure of interest rates*. From (2.1) it is clear that, for fixed t , the *interest rate implied by the yield curve* is given by

$$r(T) = r(t, T) = \frac{d}{dT} [(T - t) Y(t, T)] = -\frac{1}{V(t, T)} \frac{\partial}{\partial T} V(t, T). \quad (2.2)$$

By following arguments given in Wilmott et al. [15] extended to the present case and using no-arbitrage arguments, for the value of a zero-coupon bond $V(t, T)$, the expression

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial r^2} + v \frac{\partial V}{\partial \theta} - rV}{\partial V / \partial r},$$

must be independent of maturity, and hence,

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial r^2} + v \frac{\partial V}{\partial \theta} - rV}{\partial V / \partial r} = \varsigma(r, \theta, t), \quad (2.3)$$

for some function $\varsigma(r, \theta, t)$ independent of T . We set

$$\varsigma(r, \theta, t) = -u(r, \theta). \quad (2.4)$$

Thus, the zero-coupon bond pricing equation, providing value $V(t, T)$ of a bond at time $t < T$, in a generalized Vasicek-Malkiel model is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2} + [\eta + \gamma(\theta - r)] \frac{\partial V}{\partial r} + \mu(r - \theta) \frac{\partial V}{\partial \theta} - rV = 0. \quad (2.5)$$

We now seek a solution of (2.5) in the form:

$$V(t, T) = Ze^{A(\tau) + rB(\tau) + \theta C(\tau)}, \quad (2.6)$$

for suitable $A(\tau)$, $B(\tau)$ and $C(\tau)$ as functions of τ . Note that in order to satisfy final condition $V(T, T) = Z$, we must have

$$A(0) = 0, B(0) = 0 \text{ and } C(0) = 0.$$

Substituting (2.6) in (2.5) we obtain

$$\begin{aligned} \left(-\frac{dA}{d\tau} + \frac{1}{2}\sigma^2 B^2 + \eta B \right) + r \left(-\frac{dB}{d\tau} - \gamma B + \mu C - 1 \right) \\ + \theta \left(-\frac{dC}{d\tau} + \gamma B - \mu C \right) = 0. \end{aligned} \quad (2.7)$$

Coefficients of 1, r and θ must be zero, giving three equations for the determination of $A(\tau)$, $B(\tau)$ and $C(\tau)$:

$$\frac{dA}{d\tau} = \frac{1}{2}\sigma^2 B^2 + \eta B, \quad (2.8)$$

$$\frac{dB}{d\tau} = -\gamma B + \mu C - 1, \quad (2.9)$$

$$\frac{dC}{d\tau} = \gamma B - \mu C. \quad (2.10)$$

Note that the two equations (2.9) and (2.10) is a coupled linear inhomogeneous system for the determination of $B(\tau)$ and $C(\tau)$. Once $B(\tau)$ is known, $A(\tau)$ can be directly found from (2.8).

Now, in order to solve equations (2.9) and (2.10) for $B(\tau)$ and $C(\tau)$, we set

$$\mathbf{x}(\tau) = \begin{bmatrix} B(\tau) \\ C(\tau) \end{bmatrix}, \quad \Phi(\tau) = \begin{bmatrix} -\gamma & \mu \\ \gamma & -\mu \end{bmatrix}, \quad \mathbf{f}(\tau) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Then, equations (2.9) and (2.10) can be written as a linear inhomogeneous system:

$$\frac{d\mathbf{x}}{d\tau} = \Phi(\tau) \mathbf{x} + \mathbf{f}(\tau), \quad (2.11)$$

subject to the initial condition $\mathbf{x}(0) = \mathbf{0}$. If $\Psi(\tau)$ is a fundamental matrix for the linear homogeneous system in (2.11), then the solution of the linear inhomogeneous system (2.11) is given by

$$\mathbf{x}(\tau) = \Psi(\tau) \left[\Psi(0)^{-1} \mathbf{x}(0) + \int_0^\tau \Psi(s)^{-1} \mathbf{f}(s) ds \right]. \quad (2.12)$$

To find $\Psi(\tau)$ we need to calculate the eigenvalues and eigenvectors of $\Phi(\tau)$. The eigenvalues λ of $\Phi(\tau)$ are given by

$$\lambda^2 + (\mu + \gamma) \lambda = 0,$$

giving

$$\lambda_1 = 0, \quad \lambda_2 = -(\mu + \gamma).$$

For $\lambda_1 = 0$, an eigenvector is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \gamma/\mu \end{bmatrix},$$

while for $\lambda_2 = -(\mu + \gamma)$, an eigenvector is

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Then, a fundamental matrix for the linear homogeneous system in (2.11) is given by

$$\Psi(\tau) = \begin{bmatrix} 1 & e^{-(\mu+\gamma)\tau} \\ \gamma/\mu & -e^{-(\mu+\gamma)\tau} \end{bmatrix},$$

with its inverse

$$\Psi(\tau)^{-1} = \frac{\mu}{\mu + \gamma} \begin{bmatrix} 1 & 1 \\ \frac{\gamma}{\mu} e^{(\mu+\gamma)\tau} & -e^{(\mu+\gamma)\tau} \end{bmatrix}.$$

The solution of the linear inhomogeneous system (2.11) can now be calculated from (2.12) and is given by

$$\begin{aligned} \mathbf{x}(\tau) &= \begin{bmatrix} 1 & e^{-(\mu+\gamma)\tau} \\ \gamma/\mu & -e^{-(\mu+\gamma)\tau} \end{bmatrix} \left\{ \frac{1}{\mu + \gamma} \begin{bmatrix} -\mu\tau \\ -\frac{\gamma}{\mu+\gamma} (e^{(\mu+\gamma)\tau} - 1) \end{bmatrix} \right\} \\ &= \frac{1}{\mu + \gamma} \begin{bmatrix} -\mu\tau - \frac{\gamma}{\mu+\gamma} (1 - e^{-(\mu+\gamma)\tau}) \\ -\gamma\tau + \frac{\gamma}{\mu+\gamma} (1 - e^{-(\mu+\gamma)\tau}) \end{bmatrix}. \end{aligned}$$

Then, the solution of equations (2.9) and (2.10) is given by

$$B(\tau) = -\frac{1}{\mu + \gamma} [\mu\tau + \gamma P(\tau)], \quad (2.13)$$

and

$$C(\tau) = -\frac{\gamma}{\mu + \gamma} [\tau - P(\tau)], \quad (2.14)$$

where we have set

$$P(\tau) = \frac{1 - e^{-(\mu + \gamma)\tau}}{\mu + \gamma}. \quad (2.15)$$

We can now calculate $A(\tau)$ from (2.8) as

$$A(\tau) = \eta I_1(\tau) + \frac{1}{2} \sigma^2 I_2(\tau), \quad (2.16)$$

where, since $A(0) = 0$, we have set

$$I_1(\tau) = \int_0^\tau B(s) ds, \quad I_2(\tau) = \int_0^\tau (B(s))^2 ds.$$

For the purpose of evaluating $I_1(\tau)$ and $I_2(\tau)$ we calculate:

$$\int P(s) ds = \frac{1}{\mu + \gamma} \left[s + \frac{1}{\mu + \gamma} - P(s) \right],$$

therefore

$$\int_0^\tau P(s) ds = -\frac{1}{\gamma} C(\tau).$$

Again, integrating by parts we get

$$\int sP(s) ds = \frac{1}{\mu + \gamma} \left[\frac{s^2}{2} + \frac{s}{\mu + \gamma} + \frac{1}{(\mu + \gamma)^2} - \left(s + \frac{1}{\mu + \gamma} \right) P(s) \right],$$

therefore

$$\int_0^\tau sP(s) ds = \frac{1}{\mu + \gamma} \left[\frac{\tau^2}{2} - \frac{1}{\gamma} C(\tau) - \tau P(\tau) \right].$$

Finally, we calculate

$$\begin{aligned} \int (P(s))^2 ds &= \left[\frac{1}{(\mu + \gamma)^2} s + \frac{2}{\mu + \gamma} - 2P(s) \right. \\ &\quad \left. - \frac{1}{2(\mu + \gamma)} \{(\mu + \gamma)P(s) - 1\}^2 \right], \end{aligned}$$

then

$$\int_0^\tau (P(s))^2 ds = -\frac{1}{\mu + \gamma} \left[\frac{1}{\gamma} C(\tau) + \frac{1}{2} (P(\tau))^2 \right].$$

Note that this implies $C(\tau)$ must be negative. With the help of the above results we get

$$\begin{aligned} I_1(\tau) &= -\frac{1}{\mu + \gamma} \int_0^\tau [\mu s + \gamma P(s)] ds \\ &= -\frac{1}{\mu + \gamma} \left[\frac{1}{2} \mu \tau^2 - C(\tau) \right], \end{aligned} \quad (2.17)$$

and

$$\begin{aligned} I_2(\tau) &= \frac{1}{(\mu + \gamma)^2} \int_0^\tau [\mu^2 s^2 + 2\mu\gamma s P(s) + \gamma^2 (P(s))^2] ds \\ &= \frac{1}{(\mu + \gamma)^3} \left[\mu\gamma\tau^2 + \frac{\mu^2}{3} (\mu + \gamma) \tau^3 - (2\mu + \gamma) C(\tau) \right. \\ &\quad \left. - 2\mu\gamma\tau P(\tau) - \frac{\gamma^2}{2} (P(\tau))^2 \right]. \end{aligned} \quad (2.18)$$

Substituting for $I_1(\tau)$ and $I_2(\tau)$, from (2.16) we obtain

$$\begin{aligned} A(\tau) &= -\frac{\eta}{\mu + \gamma} \left[\frac{1}{2} \mu \tau^2 - C(\tau) \right] \\ &+ \frac{1}{2} \frac{\sigma^2}{(\mu + \gamma)^3} \left[\mu\gamma\tau^2 + \frac{\mu^2}{3} (\mu + \gamma) \tau^3 - (2\mu + \gamma) C(\tau) \right. \\ &\quad \left. - 2\mu\gamma\tau P(\tau) - \frac{\gamma^2}{2} (P(\tau))^2 \right]. \end{aligned} \quad (2.19)$$

Thus, the price of a zero-coupon bond by the generalized Vasicek-Malkiel model is given by (2.6) with $A(\tau)$, $B(\tau)$ and $C(\tau)$ given, respectively, by (2.19), (2.13) and (2.14).

The yield curve for the generalized Vasicek-Malkiel bond pricing model is given by

$$\begin{aligned} Y(t, T) &= -\frac{1}{\tau} \ln \left(\frac{V(t, T)}{V(T, T)} \right) \\ &= -\frac{1}{\tau} [A(\tau) + rB(\tau) + \theta C(\tau)]. \end{aligned} \quad (2.20)$$

To know the long-term behavior of the yield curve we calculate, as $\tau \rightarrow \infty$,

$$\frac{A(\tau)}{\tau} \sim -\frac{\eta}{\mu + \gamma} \left[\frac{\gamma}{\mu + \gamma} + \frac{1}{2}\mu\tau \right] + \frac{1}{2} \frac{\sigma^2}{(\mu + \gamma)^3} \frac{(2\mu + \gamma)\gamma}{\mu + \gamma} + \mu\gamma\tau + \frac{\mu^2}{3} (\mu + \gamma) \tau^2 \Big],$$

$$\frac{B(\tau)}{\tau} \sim -\frac{\mu}{\mu + \gamma},$$

$$\frac{C(\tau)}{\tau} \sim -\frac{\gamma}{\mu + \gamma},$$

therefore,

$$Y(t, T) \sim \frac{\eta}{\mu + \gamma} \left[\frac{\gamma}{\mu + \gamma} + \frac{1}{2}\mu\tau \right] - \frac{1}{2} \frac{\sigma^2}{(\mu + \gamma)^3} \left[\frac{(2\mu + \gamma)\gamma}{\mu + \gamma} + \mu\gamma\tau + \frac{\mu^2}{3} (\mu + \gamma) \tau^2 \right] + \frac{1}{\mu + \gamma} [\mu r + \gamma\theta]. \quad (2.21)$$

We next consider the following two particular cases of the generalized Vasicek-Malkiel model.

The Case $\eta = 0$. So far we have treated η in the short rate model (1.3) as a constant. In fact, we have introduced η for the purpose of treating it as a function of t in order to fit today's term structure of interest rates into the model which we do in the next section. Here, we can set $\eta = 0$ to simplify the present generalized Vasicek-Malkiel model.

For $\eta = 0$, $P(\tau)$, $B(\tau)$ and $C(\tau)$ remain the same as obtained above; setting $\eta = 0$ only simplifies $A(\tau)$ which is now given by

$$A(\tau) = \frac{1}{2} \frac{\sigma^2}{(\mu + \gamma)^3} \left[\mu\gamma\tau^2 + \frac{\mu^2}{3} (\mu + \gamma) \tau^3 - (2\mu + \gamma) C(\tau) - 2\mu\gamma\tau P(\tau) - \frac{\gamma^2}{2} (P(\tau))^2 \right],$$

and the asymptotic yield curve is

$$Y(t, T) \sim -\frac{1}{2} \frac{\sigma^2}{(\mu + \gamma)^3} \left[\frac{(2\mu + \gamma)\gamma}{\mu + \gamma} + \mu\gamma\tau + \frac{\mu^2}{3} (\mu + \gamma) \tau^2 \right] + \frac{1}{\mu + \gamma} [\mu r + \gamma\theta].$$

The Case $\mu = 0$. For $\mu = 0$, the short rate model (1.3) is identical with the Vasicek model for the short rate (1.1), and zero-coupon bond price is given by

$$V(t, T) = Ze^{A(\tau) + rB(\tau)},$$

where now

$$P(\tau) = \frac{1 - e^{-\gamma\tau}}{\gamma}, \quad B(\tau) = -P(\tau),$$

and $A(\tau)$ simplifies to

$$A(\tau) = \frac{1}{\gamma} \left(\frac{1}{2} \frac{\sigma^2}{\gamma} - \eta \right) (\tau - P(\tau)) - \frac{\sigma^2}{4\gamma} (P(\tau))^2,$$

while the asymptotic yield curve is

$$Y(t, T) \sim \frac{1}{\gamma} \left(\eta - \frac{1}{2} \frac{\sigma^2}{\gamma} \right).$$

3. Incorporating Initial Yield in the Generalized Vasicek-Malkiel Model

In order to incorporate today's yield into the generalized Vasicek-Malkiel model we now treat η as function of time $\eta(t)$. From (2.8) we then have

$$\frac{dA}{dt} = -\frac{1}{2}\sigma^2 (B(t, T))^2 - \eta(t) B(t, T).$$

Integrating from T to t :

$$A(t, T) = \frac{1}{2}\sigma^2 I_2(\tau) + \int_t^T \eta(s) B(T-s) ds, \quad (3.1)$$

where

$$\int_t^T (B(s, T))^2 ds = \int_0^\tau (B(s))^2 ds = I_2(\tau).$$

Suppose we wish to fit the initial yield known from the market at $t = 0$. Then, from (2.20) we have

$$Y_0(T) = -\frac{1}{T} [A_0(T) + r_0 B_0(T) + \theta_0 C_0(T)]. \quad (3.2)$$

where we have set

$$\begin{aligned} A_0(T) &= A(0, T), \quad B_0(T) = B(0, T), \quad C_0(T) = C(0, T), \\ r_0 &= r(0), \quad \theta_0 = \theta(0), \quad \text{and } Y_0(T) = Y(0, T). \end{aligned}$$

Substituting for $A(0, T)$ from (3.2) in (3.1) we get

$$\int_0^T \eta(s) B(T-s) ds = -F_0(T), \quad (3.3)$$

where we have set

$$F_0(T) = TY_0(T) + r_0B_0(T) + \theta_0C_0(T) + \frac{1}{2}\sigma^2I_2(T). \quad (3.4)$$

and from (2.18),

$$\begin{aligned} I_2(T) &= \frac{1}{(\mu + \gamma)^3} \left[\mu\gamma T^2 + \frac{\mu^2}{3}(\mu + \gamma)T^3 - (2\mu + \gamma)C(T) \right. \\ &\quad \left. - 2\mu\gamma TP(T) - \frac{\gamma^2}{2}(P(T))^2 \right]. \end{aligned}$$

Note that $F_0(T)$ is known at time $t = 0$ and that $F_0(0) = 0$. Note also that (3.3) is an integral equation for the determination of η , and once η is known, $A(t, T)$ can be found from (3.1).

Now, to determine $\eta(s)$ from (3.3), substituting for $B(T-s)$ from (2.13) and for $P(T-s)$ from (2.15), equation (3.3) can be written as

$$\int_0^T \eta(s) \left[\gamma + \mu(\mu + \gamma)(T-s) - \gamma e^{-(\mu+\gamma)(T-s)} \right] ds = (\mu + \gamma)^2 F_0(T). \quad (3.5)$$

Differentiating with respect to T and using the following rule of differentiation:

$$\frac{d}{dT} \int_0^T \eta(s) g(T-s) ds = \int_0^T \eta(s) \frac{\partial}{\partial T} g(T-s) ds + \eta(T) g(0), \quad (3.6)$$

we obtain

$$\int_0^T \eta(s) \left[\mu + \gamma e^{-(\mu+\gamma)(T-s)} \right] ds = (\mu + \gamma) F'_0(T). \quad (3.7)$$

Adding (3.5) and (3.7) we have

$$\int_0^T \eta(s) [1 + \mu(T-s)] ds = (\mu + \gamma) F_0(T) + F'_0(T). \quad (3.8)$$

Differentiating again with respect to T and using (3.6):

$$\mu \int_0^T \eta(s) ds + \eta(T) = (\mu + \gamma) F_0' T + F_0''(T). \quad (3.9)$$

Differentiating with respect to T once more we get

$$\eta'(T) + \mu \eta(T) = (\mu + \gamma) F_0''(T) + F_0'''(T). \quad (3.10)$$

This is a first order ordinary differential equation for the determination of $\eta(T)$.

Now, integrating (3.10) from 0 to s with the integrating factor $e^{\mu T}$, we obtain

$$e^{\mu s} \eta(s) = \eta(0) + (\mu + \gamma) \int_0^s e^{\mu u} F_0''(u) du + \int_0^s e^{\mu u} F_0'''(u) du.$$

Evaluating the second integral by parts we get

$$\begin{aligned} \eta(s) = & e^{-\mu s} \eta(0) + [F_0''(s) - e^{-\mu s} F_0''(0)] \\ & + \gamma e^{-\mu s} \int_0^s e^{\mu u} F_0''(u) du. \end{aligned} \quad (3.11)$$

Substituting for $\eta(s)$ from (3.11) in (3.1) we have

$$\begin{aligned} A(t, T) = & \frac{1}{2} \sigma^2 I_2(\tau) + [\eta(0) - F_0''(0)] J_1(t, T) \\ & + J_2 + \gamma J_3, \end{aligned} \quad (3.12)$$

where we have set

$$\begin{aligned} J_1(t, T) &= \int_t^T e^{-\mu s} B(T-s) ds, \quad J_2 = \int_t^T B(T-s) F_0''(s) ds, \\ J_3 &= \int_t^T e^{-\mu s} B(T-s) \left(\int_0^s e^{\mu u} F_0''(u) du \right) ds. \end{aligned}$$

Calculating these integrals, we get

$$J_1(t, T) = \frac{1}{\gamma} e^{-\mu t} C(\tau).$$

For J_3 , changing the order of integration we have

$$\begin{aligned} J_3 &= J_1(t, T) K(t) + \int_t^T e^{\mu u} F_0''(u) J_1(u, T) du, \\ &= J_1(t, T) K(t) + \frac{1}{\gamma} \int_t^T C(T-u) F_0''(u) du, \end{aligned}$$

where we have set

$$K(t) = \int_0^t e^{\mu u} F_0''(u) du.$$

Then,

$$\begin{aligned} J_2 + \gamma J_3 &= e^{-\mu t} C(\tau) K(t) + \int_t^T [B(T-s) + C(T-s)] ds \\ &= e^{-\mu t} C(\tau) K(t) - \int_t^T (T-s) F_0''(s) ds \\ &= e^{-\mu t} C(\tau) K(t) - [F_0(T) - F_0(t)] + F_0'(t) \tau. \end{aligned}$$

With these results, from (3.12) we finally obtain

$$\begin{aligned} A(t, T) &= \frac{1}{2} \sigma^2 I_2(\tau) + \frac{1}{\gamma} e^{-\mu t} C(\tau) [\eta(0) - F_0''(0)] \\ &\quad + e^{-\mu t} C(\tau) K(t) - [F_0(T) - F_0(t)] + F_0'(t) \tau. \end{aligned} \quad (3.13)$$

Note that $F_0(t)$ is given by (3.4).

To be able to use this expression for $A(t, T)$ we need to simplify it as far as possible. For the purpose we first define *forward yield* at $t = 0$ by

$$f(t, T) = \frac{Y_0(T)T - Y_0(t)t}{T - t}.$$

Then,

$$\begin{aligned} F_0(T) - F_0(t) &= f(t, T) \tau + r_0 B^*(t, T) + \theta_0 C^*(t, T) \\ &\quad + \frac{1}{2} \sigma^2 \{I_2(T) - I_2(t)\}, \end{aligned}$$

where we have set

$$\begin{aligned} B^*(t, T) &= B_0(T) - B_0(t) = -\frac{1}{\mu + \gamma} \left[\mu \tau + \gamma e^{-(\mu + \gamma)t} P(\tau) \right], \\ C^*(t, T) &= C_0(T) - C_0(t) = -\frac{\gamma}{\mu + \gamma} \left[\mu \tau - e^{-(\mu + \gamma)t} P(\tau) \right]. \end{aligned}$$

Also,

$$F_0'(t) = r(0, t) - r_0 + \gamma(r_0 - \theta_0)P(t) + \frac{1}{2} \sigma^2 I_2'(t),$$

where

$$I_2'(t) = \frac{1}{(\mu + \gamma)^2} [\mu^2 t^2 + \gamma P(t) \{2\mu t + \gamma P(t)\}],$$

and

$$F_0''(t) = \frac{d}{dt}r(0, t) + \gamma(r_0 - \theta_0)e^{-(\mu+\gamma)t} + \frac{1}{2}\sigma^2 I_2''(t),$$

where

$$I_2''(t) = \frac{2}{\mu + \gamma} [\mu t + \gamma P(t) \{1 - \gamma P(t)\}].$$

Note that for $t = 0$, since $I_2(0) = I_2'(0) = I_2''(0) = 0$, and since from the stochastic differential equation (1.3),

$$\eta(0) = \left. \frac{d}{dt}r(0, t) \right|_{t=0} + \gamma(r_0 - \theta_0),$$

therefore,

$$\eta(0) - F_0''(0) = \eta(0) - \left[\left. \frac{d}{dt}r(0, t) \right|_{t=0} + \gamma(r_0 - \theta_0) \right] = 0.$$

Again, substituting for $F_0''(u)$ in $K(t)$, we have

$$K(t) = M(t) + (r_0 - \theta_0)(1 - e^{-\mu t}) + \frac{1}{2}\sigma^2 N(t),$$

where we have set

$$\begin{aligned} N(t) &= \int_0^t e^{\mu u} I_2''(u) du = \frac{2}{\mu + \gamma} \left[t e^{\mu t} - \frac{\mu^2 + \mu\gamma + \gamma^2}{\mu(\mu + \gamma)^2} (e^{\mu t} - 1) \right. \\ &\quad \left. + \frac{\mu - \gamma}{(\mu + \gamma)^2} (e^{-\gamma t} - 1) + \frac{\gamma^2}{(\mu + \gamma)^2(\mu + 2\gamma)} (e^{-(\mu+2\gamma)t} - 1) \right], \end{aligned}$$

and

$$\begin{aligned} M(t) &= \int_0^t e^{\mu u} \frac{d}{du}r(0, u) du \\ &= e^{\mu t}r(0, t) - r_0 - \mu Q(t), \end{aligned}$$

where

$$Q(t) = \int_0^t e^{\mu u} r(0, u) du.$$

With these results from (3.13) we finally obtain

$$A(t, T) = \frac{1}{2}\sigma^2 I_2(\tau)$$

$$\begin{aligned}
& + e^{-\mu t} C(\tau) \left[M(t) + (r_0 - \theta_0) (1 - e^{-\mu t}) + \frac{1}{2} \sigma^2 N(t) \right] \\
& - \left[f(t, T) \tau + r_0 B^*(t, T) + \theta_0 C^*(t, T) + \frac{1}{2} \sigma^2 \{I_2(T) - I_2(t)\} \right] \\
& + \tau \left[r(0, t) - r_0 + \gamma (r_0 - \theta_0) P(t) + \frac{1}{2} \sigma^2 I_2'(t) \right]. \quad (3.14)
\end{aligned}$$

With this $A(t, T)$, the value of a zero-coupon bond by the generalized Vasicek-Malkiel model, with the initial term structure of interest rates incorporated at time $t = 0$, is given by (2.6) where the values of $B(t, T)$ and $C(t, T)$ are as given before. While from (1.4) we can express $\mu Q(t) = e^{\mu t} [\theta(t) - \theta_0]$, $Q(t)$ can better be estimated numerically by using past short rates for the time 0 up to t when we want to price a bond $V(t, T)$.

Note that for $t = 0$, (3.14) reduces to

$$A(0, T) = -[Y_0(T)T + r_0 B_0(T) + \theta_0 C_0(T)],$$

as it should, while for $t = T$, $A(T, T) = 0$. We also note the following particular case.

The Case $\mu = 0$. For $\mu = 0$, the short rate model (1.3) is identical with the Vasicek model for the short rate (1.1), and the price of a zero-coupon bond is given by

$$V(t, T) = Z e^{A(\tau) + rB(\tau)},$$

where now

$$\begin{aligned}
P(\tau) &= \frac{1 - e^{-\gamma\tau}}{\gamma}, \quad B(\tau) = -P(\tau), \quad C(\tau) = -\tau + P(\tau), \\
M(t) &= r(0, t) - r_0, \quad N(t) = I_2'(t), \quad B^*(t, T) = -e^{-\gamma t} P(\tau),
\end{aligned}$$

and $A(t, T)$ in (3.14) simplifies to

$$\begin{aligned}
A(t, T) &= \frac{1}{2} \sigma^2 [I_2(\tau) - I_2(T) + I_2(t)] - f(t, T) \tau \\
&+ \left[r(0, t) - r_0 (1 - e^{-\gamma t}) + \gamma r_0 \tau + \frac{1}{2} \sigma^2 I_2'(t) \right] P(\tau).
\end{aligned}$$

References

- [1] F. Black, E. Derman and W. Toy, A one-factor model of interest rates and its application to treasury bond options, *Financial Analysts Journal*, **46** (1990), 33-39.
- [2] M.M. Chawla, On solutions of the bond pricing equation, *Intern. J. Appl. Math.*, **23** (2010), 661-680.
- [3] J.C. Cox, J.E. Ingersoll and S.A. Ross, A re-examination of traditional hypotheses about the term structure of interest rates, *Journal of Finance*, **36** (1981), 769-799.
- [4] D. Duffie and R. Kan, A yield-factor model of interest rates, *Mathematical Finance*, **64** (1996), 379-406.
- [5] D. Heath, R. Jarrow and A. Morton, Bond pricing and the term structure of interest rates: A new methodology, *Econometrica*, **60** (1992), 77-105.
- [6] T.S. Ho and S.B. Lee, Term structure movements and pricing interest rate contingent claims, *Journal of Finance*, **41** (1986), 1129-1142.
- [7] L. Hughston, editor, *Vasicek and Beyond: Approaches to Building and Applying Interest Rate Models*, Risk Books, London, 1997.
- [8] J.C. Hull, *Options, Futures and Other Derivative Securities*, sixth edition, Prentice-Hall, New York, 2005.
- [9] J. Hull and A. White, Pricing interest rate derivative securities, *Review of Financial Studies*, **3** (1990), 573-592.
- [10] J. Hull and A. White, One-factor interest rate models and the valuation of interest rate derivative securities, *Journal of Financial and Quantitative Analysis*, **28** (1993), 235-254.
- [11] R. Klugman, Interest rate modelling, *OCIAM working paper*, Mathematical Institute, Oxford University, 1992.
- [12] B. Malkiel, *The Term Structure of Interest Rates: Expectations and Behavior Patterns*, Princeton University Press, Princeton, 1966.
- [13] R.S. Mamon, Three ways to solve for bond prices in the Vasicek model, *Journal of Applied Mathematics and Decision Sciences*, **8** (2004), 1-14.

- [14] O. Vasicek, An equilibrium characterization of the term structure, *Journal of Financial Economics*, **5** (1977), 177-188.
- [15] P. Wilmott, S. Howison and J. Dewynne, *The Mathematics of Financial Derivatives: A Student Introduction*, Cambridge University Press, Cambridge, 1995.