OPTIMIZATION OF SPLITTING POSITIVE DEFINITE MIXED FINITE ELEMENT METHODS

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Abstract: In this paper, we investigate error analysis of splitting positive definite mixed finite element methods for optimal control problems with an control constraints. The finite element methods are discretized by the Raviart-Thomas mixed space and the control variable is approximated by piecewise polynomials. Finally we derive error analysis for both the control variable and the state variables, when the control is discretized by piecewise linear continuous functions and illustrate with a numerical example to confirm our theoretical results.

1. Introduction

Optimal control problems are playing an increasingly important role in science and engineering. In many optimal control problems, the objective functional contains not only the primal state variable but also its gradient. The advantage of mixed element methods is that the approximations to u and the flux $\boldsymbol{p}$ can be obtained simultaneously. Many researchers have made various contributions to the mixed finite element methods and adaptive finite element method for optimal control problems found in [2, 3, 4, 5, 8, 11, 12]. The advantage of this

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method is that the approximations to the unknown variable and its flux can be obtained simultaneously. There are some research articles on mixed finite element methods for optimal control problems [9, 10, 15, 17, 19]. However, the technique of the classical mixed finite element methods leads to some saddle point problems which are difficult to solve numerically because of losing positive definite properties. Moreover it usually needs to solve a coupled system of equations which brings in difficulties to some extent. Thus the popular preconditioned conjugate gradient solvers cannot be used for the solution of linear algebraic systems.

Recently, Guo in [13], the author has established a new mixed finite element method to solve the second-order hyperbolic and pseudo-hyperbolic integro-differential equations, in which the mixed element system is symmetric positive definite without requiring the Ladyzhenskaya-Babuska-Brezzi consistency condition. Guo et al. in [14], the author has discussed two novel mixed finite elements for parabolic integro-differential equations which can be split into two independent symmetric positive definite subschemes and do not need to solve a coupled system of equations without requiring the Ladyzhenskaya-Babuska-Brezzi consistency condition. Further convergence analysis, shows that both $L^2(\Omega)$- norm error estimate for u and optimal $H(\text{div};\Omega)$- norm error estimate for σ. Liu et al. in [18], the author has used splitting positive definite mixed finite element methods for a class of second-order pseudo-hyperbolic equations. Error estimates are derived for both semi-discrete and fully discrete schemes are proved.

Yang in [22], the author has studied a miscible displacement of one compressible fluid by another in a porous medium governed by a nonlinear parabolic system. A new mixed finite element method in which the mixed element system is symmetric positive definite and the flux equation is separated from pressure equation is used to solve the pressure equation of parabolic type and a standard Galerkin method is used to treat the convection-diffusion equation of concentration of one of the fluids. The convergence of the approximate solution with an optimal accuracy in L^2-norm is proved. Zhang et al. in [23], the author has established a new mixed finite element procedure in which the mixed element system is symmetric positive definite to solve the second-order hyperbolic equations. Error estimates and convergence of the mixed element methods with continuous-time and discrete-time scheme are proved. Wang et al. in [21], the author has presented a splitting positive definite mixed finite element procedure to solve the second-order hyperbolic equation and further analyzed the superconvergence property of the mixed element methods with discrete-time approximation for the hyperbolic equation. In this work, we develop a priori error

analysis of splitting positive definite mixed finite element methods for quadratic optimal control problems governed by elliptic partial differential equations. Our resulting procedure for the control constraint consists of splitting positive symmetric definite matrix for whole domain by mixed finite element methods seen new and then we use piecewise linear polynomial functions to approximate the control variable by preconditioning projection gradient method respectively.

In this paper, we study splitting positive definite system mixed finite element methods for optimal control problems governed by elliptic partial differential equations:

$$\min_{u \in K \subset L^2(\Omega)} \left\{ \frac{1}{2} \Big(\|p - p_d\|^2 + \|y - y_d\|^2 + \alpha \|u\|^2 \Big) \right\}, \tag{1.1}$$

$$\operatorname{div} p = f + u, x \in \Omega, \tag{1.2}$$

$$p = -A\nabla y, x \in \Omega, \tag{1.3}$$

$$y = 0, \ x \in \partial\Omega, \tag{1.4}$$

where $\Omega \in \mathbb{R}^2$ is a bounded open set with the boundary $\partial\Omega$, $\alpha > 0$, $p_d \in (H^1(\Omega))^2$ and $y_d \in L^2(\Omega)$. Here K denotes the admissible set of the control variable defined by

$$K = \left\{ u \in L^2(\Omega) : \int_\Omega u \geq 0 \right\}. \tag{1.5}$$

Throughout this paper, for the splitting positive definite mixed finite element methods for optimal control problems, it is proved that these approximations have convergence order $O(h^{k+1})$. In Section 2, we formulate the splitting positive definite mixed finite element methods approximation for optimal control problems. In Section 3, existence of the control variables are stated. Then we study a priori error analysis for the state, costate and control variables in Section 4. Finally we illustrate with a numerical example to confirm our theoretical results in Section 5.

2. Notations and Preliminaries

The following notation will be used throughout the article:

$$
\begin{aligned}
\mathcal{T}_h &= \text{a regular simplicial triangulations of } \Omega \\
T &= \text{a triangle of } \mathcal{T}_h \\
\rho(T) &= \text{diameter of the set } T \\
\sigma(T) &= \text{diameter of the largest ball contained in } T \\
h &= \max\{\rho(T) : T \in \mathcal{T}_h\}.
\end{aligned}
$$

Define the inner products:

$$(y, v) = \int_\Omega yv dx, \quad \forall y, v \in L^2(\Omega), \tag{2.1}$$

$$(p, v) = \sum_{i=1}^{2}(p_i, v_i), \quad \forall p, v \in (L^2(\Omega))^2. \tag{2.2}$$

Furthermore, [1] For $1 \le p \le \infty$ and m any nonnegative integer, we consider three vector spaces on which $\| \cdot \|_{m,p}$ is a norm:

(a) $H^{m,p}(\Omega) \equiv$ the completion of $\{\phi \in \mathbb{C}^m(\Omega) : \|\phi\|_{m,p} < \infty\}$ with respect to the norm $\| \cdot \|_{m,p}$.

(b) $W^{m,p}(\Omega) \equiv \{\phi \in L^p(\Omega) : D^\alpha \phi \in L^p(\Omega) \ for \ 0 \le |\alpha| \le m\}$, where $D^\alpha \phi$ is the weak partial derivative of ϕ, then sobolev norm is given by

$$\|\phi\|_{m,p} = \Big(\sum_{0 \le |\alpha| \le m} \|D^\alpha \phi\|_{L^p(\Omega)}^p \Big)^{1/p}, \quad \|\phi\|_{m,p} = \max_{0 \le |\alpha| \le m} \|D^\alpha \phi\|_\infty$$

and the semi-norm $| \cdot |_{m,p}$ given by

$$|\phi|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha \phi\|_{L^p(\Omega)}^p.$$

We set $W_0^{m,p}(\Omega) = \{\phi \in W^{m,p}(\Omega) : \phi \mid_{\partial\Omega}= 0\}$. For any m, we have the obvious chain of imbeddings

$$W_0^{m,p}(\Omega) \to W^{m,p}(\Omega) \to L^p(\Omega).$$

For $p = 2$, we denote $H^{m,p}(\Omega) = W^{m,2}(\Omega), H_0^{m,p}(\Omega) = W_0^{m,2}(\Omega), \|.\|_m = \|.\|_{m,2}$ and $\|.\| = \|.\|_{0,2}$.

(d) We assume that $A(x) = (a_{ij}(x))_{2\times2}$ is a bounded symmetric and positive definite matrix with $a_{ij}(x) \in \mathbb{C}^\infty(\Omega)$ and, for any vector $\xi \in \mathbb{R}^2$, there is a constant C>0 such that $\xi^T A\xi \ge C\|\xi\|_{\mathbb{R}^2}^2$.

(e) $C = C(x)$ is positive definite and bounded, that is, there exist positive constants C_1 and C_2 such that $0 < C_1 \le C \le C_2$.

Define a weak formulation of the problem (1.1)-(1.4). Let

$$V = H(\text{div}; \Omega) = \{\text{v} \in (L^2(\Omega))^2, \text{divv} \in L^2(\Omega)\}.$$

The Hilbert space V is equipped with the following norm given by

$$\|v\|_{H(\text{div};\Omega)} = (\|v\|^2 + \|\text{div}v\|^2)^{1/2} \text{ and } W = L^2(\Omega).$$

In order to consider splitting positive definite mixed finite element approximation of the elliptic optimal control problem (1.1)-(1.4), we find $(p, y, u) \in V \times W \times K$ such that

$$\underset{u \in K \subset L^2(\Omega)}{\text{minimize}} \left\{ \frac{1}{2} \left(\|p - p_d\|^2 + \|y - y_d\|^2 + \alpha\|u\|^2 \right) \right\} \tag{2.3}$$

subject to

$$(A^{-1}p, v) - (y, \text{div}v) = 0, \forall v \in V, \tag{2.4}$$
$$(\text{div}p, w) = (f, w) + (u, w), \forall w \in W. \tag{2.5}$$

From (2.4) and the boundary conditions, we have

$$(A^{-1}p, v) = (y, \text{div}v), \forall v \in V.$$

To derive a symmetric and positive definite weak formulation for problem (1.1)-(1.4), taking $w = \text{div}\tau, \forall \tau \in V$ in (2.5), we derive an equivalent mixed variational form

$$\underset{u \in K \subset L^2(\Omega)}{\text{minimize}} \left\{ \frac{1}{2} \left(\|p - p_d\|^2 + \|y - y_d\|^2 + \alpha\|u\|^2 \right) \right\}, \tag{2.6}$$
$$(y, v) + (\text{div}p, v) = (f, v) + (u, v), \forall v \in V, \tag{2.7}$$
$$(\text{div}p, \text{div}\tau) = (f, \text{div}\tau) + (u, \text{div}\tau), \forall \tau \in W. \tag{2.8}$$

It is well known that the optimal control problem (2.6)-(2.8) has a unique solution (p, y, u) and that a triple (p, y, u) is the solution if and only if there is a costate $(q, z) \in V \times W$ such that (p, y, q, z, u) for all $v \in W$, $\tau \in V$ and $u \in K \subset L^2(\Omega)$ satisfies the following optimality conditions [16]:

$$(y, v) + (\text{div}p, v) = (f, v) + (u, v), \tag{2.9}$$
$$(\text{div}p, \text{div}\tau) = (f, \text{div}\tau) + (u, \text{div}\tau), \tag{2.10}$$
$$(A^{-1}z, v) = -(y - y_d, v), \tag{2.11}$$
$$(\text{div}q, \text{div}\tau) = -(p - p_d, \tau) - (\text{div}\tau, z), \tag{2.12}$$
$$(\alpha u + z - \text{div}q, u - u^*)_K \geq 0. \tag{2.13}$$

Let $\mathcal{T}_{h_p}$ and $\mathcal{T}_{h_y}$ be two families of finite element partitions of the domains Ω which are identical or not. Let h_p and h_y be mesh parameters respectively.

Based on $\mathcal{T}_{h_p}$ and $\mathcal{T}_{h_y}$, we construct the finite element spaces $V_h \subset V$ and $W_h \subset W$ with the following approximation properties:

$$\inf_{v_h \in V_h} \|v - v_h\| \leq C h_p^{k+1} \|v\|_{(H^{k+1}(\Omega))^2}, \tag{2.14}$$

$$\inf_{v_h \in V_h} \|\mathrm{div}(v - v_h)\| \leq C h_p^{k_1} \|v\|_{(H^{k_1+1}(\Omega))^2}, \tag{2.15}$$

$$\inf_{w_h \in W_h} \|w - w_h\| \leq C h_y^{k+1} \|w\|_{H^{k+1}(\Omega)}, \tag{2.16}$$

for $\forall v \in V \cap (H^{k+1}(\Omega))^2$, $w \in W \cap H^{k+1}(\Omega)$. It is clear that we have $k_1 = k$ at least and $k_1 = k + 1$. Let V_h be selected as the Raviart-Thomas mixed finite element space of index k [20]. We define another finite element space $W_h \subset W = L^2(\Omega)$ consisting of piecewise polynomials of order $k = 1$ on each elements of $\mathcal{T}_h$. Let P_k denote the space of polynomials of total degree at most k in x_1 and x_2 variables respectively, where $X = (x_1, x_2)$

$$V = P_k(T) \oplus span(XP_k(T)), \quad W(T) = P_k(T).$$

Then we define the finite dimensional spaces as follows:

$$\begin{aligned}
V_h(T) &:= \{v_h \in V : \forall\, T \in \mathcal{T}_h, v_h\,|_T \in V(T)\}, \\
W_h(T) &:= \{w_h \in W : \forall\, T \in \mathcal{T}_h, w_h\,|_T \in W(T)\}, \\
K_h(T) &:= \{u_h \in K : \forall\, T \in \mathcal{T}_h, u_h\,|_T \in W(T)\}.
\end{aligned}$$

Let K_h be a closed convex set in W_h,

$$\text{where} \quad K_h = \left\{ u_h \in L^2(\Omega) : \int_\Omega u_h \geq 0 \right\}. \tag{2.17}$$

Then the corresponding discrete splitting positive definite mixed finite element approximation of optimal control problem which will be labeled as $(OCP)_h$ is defined as follows for all $v_h \in W_h$ and $\tau_h \in V_h$:

$$\underset{u \in K}{\text{minimize}} \left\{ \frac{1}{2} \left(\|p_h - p_d\|^2 + \|y_h - y_d\|^2 + \alpha \|u_h\|^2 \right) \right\} \tag{2.18}$$

$$(y_h, v_h) + (\mathrm{div}p_h, v_h) = (f, v_h) + (u, v_h), \tag{2.19}$$

$$(\mathrm{div}p_h, \mathrm{div}\tau_h) = (f, \mathrm{div}\tau_h) + (u_h, \mathrm{div}\tau_h). \tag{2.20}$$

It is well known that the optimal control problem (2.18)-(2.20) again has a unique solution (y_h, p_h, u_h) and that a triplet (y_h, p_h, u_h) is the solution of (2.18)-(2.20), if there is a costate $(z_h, q_h) \in V \times W$ such that $(y_h, p_h, q_h, z_h, u_h)$

for all $v_h \in W_h$, $\tau_h \in V_h$ and $u_h \in K_h \subset L^2(\Omega)$ satisfies the following optimality conditions:

$$(y_h, v_h) + (\text{div} p_h, v_h) = (f, v_h) + (u, v_h), \tag{2.21}$$

$$(\text{div} p_h, \text{div}\tau_h) = (f, \text{div}\tau_h) + (u_h, \text{div}\tau_h), \tag{2.22}$$

$$(A^{-1} z_h, v_h) = -(y_h - y_d, v_h), \tag{2.23}$$

$$(\text{div} q_h, \text{div}\tau_h) = -(p_h - p_d, \tau_h) - (\text{div}\tau_h, z_h), \tag{2.24}$$

$$(\alpha u_h + z_h - \text{div} q_h, u_h - u_h^*)_K \geq 0. \tag{2.25}$$

3. Existence of Control Variables

In this section, the following two lemmas present the solutions of the control variables for variational inequalities in (2.9)-(2.13) and (2.21)-(2.25) which are very important for the error analysis and the numerical algorithm construction later.

The regularity of the optimal control for a constrained problem is quite low, say only in $H^1(\Omega)$. For example, in our priori work, we have considered the obstacle type constraint set: $K = \{u : a \leq u \leq b\}$, where a and b are real numbers.

Remark 3.1. (see [16]) Inequality (2.13) is equivalent to the following:

$$\begin{cases} \alpha u + z - \text{div} q > 0, \ u = a, \\ \alpha u + z - \text{div} q < 0, \ u = b, \\ \alpha u + z - \text{div} q = 0, \ a < u < b. \end{cases}$$

For optimal control problem with this constraint set, we have the following relationship between the control variable u and the costate variable z. For pointwise projection operator $P_K z$ from K to K_h, then

$$P_K z(x, t) = \max(a, \min(b, z(x, t) + \text{div} q)).$$

Then, the optimality condition (2.13) can be expressed as

$$u = P_K \left(\frac{z}{\alpha} + \frac{\text{div} q}{\alpha} \right).$$

Thus the gradient of u jumps along the boundary of the zero-set of z, whose location is generally unknown. Due to the special structure of our control

constraint set K is defined in (1.5). We are able to show that the optimal control problem (2.9)-(2.12) can be infinitely smooth if the initial data are so.

Lemma 3.1. *Let $(p, y, q, z, u) \in (V \times W)^2 \times K$ be the solution of $(2.9) -$ (2.12). Assume that the data functions f, y_d, p_d and the domain Ω are infinitely smooth. Then the control function $u \in C^\infty(\bar{\Omega})$.*

Proof. By applying the assumption on f and the regularity argument of elliptic problems, it is clear that $y \in H^2(\Omega)$ so that $p \in H^1(\Omega)$. It follows, from the costate equation and the assumption on y_d, p_d, that we obtain $z \in H^2(\Omega)$. Using the relationship between the control and the costate $u = \max(0, \frac{\bar{z}}{\alpha}) - \frac{z}{\alpha} + \frac{\text{div}q}{\alpha}$, $u \in H^2(\Omega)$. Thus $y \in H^4(\Omega)$, $p \in H^3(\Omega)$. By repeating the above process, we can conclude that $u \in C^\infty(\bar{\Omega})$. □

Remark 3.2. If Ω is a convex open domain with a Lipschitz boundary $\partial\Omega$ and $y_d \in L^2(\Omega), p_d \in (H^1(\Omega))^2$, then we have $u \in H^2(\Omega)$.

Now we derive error estimates for splitting positive definite mixed finite element methods for optimal control problems governed by elliptic equations with the control approximated by piecewise polynomial element of k=1. We introduce some intermediate variables $(p_h(u^*), y_h(u^*), q_h(u^*), z_h(u^*)) \in (V_h \times W_h)^2$ for all $v_h \in W_h$ and $\tau_h \in V_h$ associated with the optimal control problems u^* as follows:

$$(y_h(u^*), v_h) + (\text{div}p_h(u^*), v_h) = (f, v_h) + (u^*, v_h), \tag{3.1}$$

$$(\text{div}p_h(u^*), \text{div}\tau_h) = (f, \text{div}\tau_h) + (u^*, \text{div}\tau_h), \tag{3.2}$$

$$(A^{-1}z_h(u^*), v_h) = -(y_h(u^*) - y_d, v_h), \tag{3.3}$$

$$(\text{div}q_h(u^*), \text{div}\tau_h) = -(p_h(u^*) - p_d, \tau_h) - (\text{div}v_h(u^*), z_h). \tag{3.4}$$

For given control u^*, it is not difficult to verify that problems (3.1)-(3.4) admit unique solutions. Then, some interpolation or projection estimates are prepared without proof.

First, we define the standard $L^2(\Omega)$-projection [6] $P_h : W \to W_h$, for any $w \in W$, satisfying:

$$(w - P_hw, w_h) = 0, \quad \forall w_h \in W_h, \tag{3.5}$$

Next we recall the Fortin projection [7] $\Pi_h : V \to V_h$ which satisfies, for any $q \in V$,

$$(\text{div}(q - \Pi_hq), w_h) = 0, \quad \forall w_h \in W_h, q \in V, \tag{3.6}$$

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that is, $\mathrm{div} \circ \Pi_h = P_h \circ \mathrm{div} : V \xrightarrow{onto} W_h$ and $\mathrm{div}(I - \Pi_h)V \perp W_h$ where I denotes the identity matrix, $k = 0, 1$,

$$\|w - P_h w\| \le Ch^{k+1}|w|, \forall w \in H^1(\Omega), \tag{3.7}$$

$$\|q - \Pi_h q\| \le Ch^k|q|_1, \forall q \in (H^1(\Omega))^2, \tag{3.8}$$

$$\|\mathrm{div}(q - \Pi_h q)\| \le Ch^k|\mathrm{div}q|, \forall \mathrm{div}q \in H^1(\Omega). \tag{3.9}$$

Lemma 3.2. *Let (p_h, y_h, q_h, z_h) and $(p_h(u), y_h(u), q_h(u), z_h(u))$ be the solutions of (2.21)-(2.24) and (3.1)-(3.4) respectively. Then the following estimates hold:*

$$\|p_h - p_h(u)\| + \|y_h - y_h(u)\| \le C\|u - u_h\|_{L^2(\Omega)}, \tag{3.10}$$

$$\|q_h - q_h(u)\| + \|z_h - z_h(u)\| \le C\|u - u_h\|_{L^2(\Omega)}. \tag{3.11}$$

Proof. First we prove (3.10) and (3.11). It follows from (2.21)-(2.24) and (3.1)-(3.4). We set the following intermediate errors:

$$r_1 = p_h - p_h(u) \quad and \quad e_1 = y_h - y_h(u),$$
$$r_2 = q_h - q_h(u) \quad and \quad e_2 = z_h - z_h(u).$$

Thus we have

$$(e_1, \mathrm{v}_h) + (\mathrm{div}r_1, \mathrm{v}_h) = (u - u_h, \mathrm{v}_h), \tag{3.12}$$

$$(\mathrm{div}r_1, \mathrm{div}\tau_h) = (u - u_h, \mathrm{div}\tau_h), \tag{3.13}$$

$$(A^{-1}e_2, v_h) = -(y_h - y_h(u), v_h), \tag{3.14}$$

$$(\mathrm{div}r_2, \mathrm{div}\tau_h) = -(p_h - p_h(u), \tau_h(u)) - (\mathrm{div}v_h, e_2). \tag{3.15}$$

Note that (3.12)-(3.15) are splitting into two parts of elliptic positive defined operator. Then using the stability property of the mixed finite space, we can prove that

$$\|r_1\|_{H(div;\Omega)} + \|e_1\|_{L^2(\Omega)} \le C\|u - u_h\|_{L^2(\Omega)}, \tag{3.16}$$

$$\|r_2\|_{H(div;\Omega)} + \|e_2\|_{L^2(\Omega)} \le C\|u - u_h\|_{L^2(\Omega)}. \tag{3.17}$$

So we need to give the estimate for $\|u - u_h\|_{L^2(\Omega)}$. $\qquad \square$

4. Estimates of Optimal Control Problem

In this section, we consider the error estimate of splitting positive defined mixed finite element methods for optimal control problems.

Lemma 4.1.   Let (p, y, q, z) and $(p_h(u), y_h(u), q_h(u), z_h(u))$ be the solutions of (2.9)-(2.12) and (3.1)-(3.4) respectively. Then the following estimates hold:

$$\|p - p_h(u)\|_{H(div;\Omega)} + \|y - y_h(u)\|_{L^2(\Omega)} \leq C(h^{k+1}\|p\|_{k+2} + h^{k+1}\|y\|_{k+1}),$$

$$\|q - q_h(u)\|_{H(div;\Omega)} + \|z - z_h(u)\|_{L^2(\Omega)} \leq C(h^{k+1}\|q\|_{k+2} + h^{k+1}\|y\|_{k+1}).$$

Proof. We define some intermediate errors for the state variables:

$$\eta_1 = p - p_h(u) \quad and \quad \lambda_1 = y - y_h(u),$$
$$\eta_2 = q - q_h(u) \quad and \quad \lambda_2 = z - z_h(u).$$

Then, from (2.9)-(2.13) and (3.1)-(3.4), $v_h \in W_h$, $\tau_h \in V_h$, we obtain the following error equations:

$$(\lambda_1, v_h) + (div\eta_1, v_h) = 0, \tag{4.1}$$
$$(div\eta_1, div\tau_h) = ((u_h - u_h), div\tau_h), \tag{4.2}$$
$$(A^{-1}\lambda_2, v_h) = -(y_h - y_h(u), v_h), \tag{4.3}$$
$$(div\eta_2, div\tau_h) = -(p_h - p_h(u), \tau_h(u)) - (divv_h, \lambda_2). \tag{4.4}$$

Setting $\tau_h = Q_h p - p_h(u)$, $v_h = R_h y - y_h(u)$ respectively, we obtain

$$(\lambda_1, R_h y - y_h(u)) + (div\eta_1, R_h y - y_h(u)) = 0, \tag{4.5}$$
$$(div\eta_1, div(Q_h p - p_h(u))) = 0. \tag{4.6}$$

Using the Cauchy inequality and the assumption of elliptic positive definite mixed finite element space, we obtain

$$\|Q_h p - p_h(u)\|^2_{H(div;\Omega)} \leq C\|p - Q_h p\|^2_{H(div;\Omega)} + \|Q_h p - p_h(u)\|^2_{H(div;\Omega)}. \tag{4.7}$$

By using the triangle inequality, (3.7) and Lemma (3.2), we conclude that (4.1) is proved.

Similarly second part of Lemma (4.1) is proved easily by setting $\tau_h = Q_h w - w_h(u)$, $v_h = R_h z - z_h(u)$ respectively. We obtain

$$(A^{-1}\eta_2, R_h z - z_h(u)) = -(p_h - p_h(u), R_h z - z_h(u)), \tag{4.8}$$
$$(div\eta_2, divQ_h w - w_h(u)) = (y_h - y_h(u), Q_h w - w_h(u)) - (divR_h z - z_h(u), \lambda_2). \tag{4.9}$$

Using the Cauchy inequality, we have

$$\|R_h z - z_h(u)\|_{L^2(\Omega)} \leq C\|z - R_h z\|_{L^2}(\Omega) + C\|y - y_h(u)\|_{L^2}(\Omega). \quad (4.10)$$

Combining intermediate operator and (4.1), by using the triangle inequality, we conclude that (4.1) is proved. $\qquad\square$

Theorem 4.1. *Let (p, y, q, z, u) and $(p_h(u), y_h(u), q_h(u), z_h(u), u_h)$ be the solutions of (2.9)-(2.13) and (3.1)-(3.4) respectively. Then we have*

$$\|u - u_h\|_{L^2(\Omega)} \leq CMh^{k+1}, \qquad (4.11)$$

$$\|y - y_h\|_{L^2(\Omega)} + \|p - p_h\|_{H(div;\Omega)} \leq CMh^{k+1}, \qquad (4.12)$$

$$\|z - z_h\|_{L^2(\Omega)} + \|q - q_h\|_{H(div;\Omega)} \leq CMh^{k+1}, \qquad (4.13)$$

where $M = \|q\|_{(H^2(\Omega))^2} + \|z\|_{H^2(\Omega)} + \|u\|_{H^2(\Omega)} + \|p\|_{k+2} + \|y\|_{k+3}$.

Proof. Note that $W_h \subset W$ and $K_h \subset K$. The variational inequalities (2.13) and (2.25) imply that

$$(\alpha u + z - \mathrm{div}q, u - u_h)_K \geq 0, \quad (\alpha u_h + z_h - \mathrm{div}q_h, u_h - P_h u)_K \geq 0. \quad (4.14)$$

Hence we deduce that

$$
\begin{aligned}
\alpha\|u - u_h\|_{L^{(\Omega)}} &= (\alpha u, u - u_h) - (\alpha u_h, u - u_h) \\
&= (\alpha u + z - \mathrm{div}q, u - u_h) + (\alpha u_h + z_h - \mathrm{div}q_h, u_h - P_h u) \\
&\quad + (\alpha u_h + z_h - \mathrm{div}q_h, P_h u - u) + (z - z_h, u - u_h) \\
&\quad + (\mathrm{div}(q - q_h), u - u_h) \\
&\leq (\alpha(u_h - u), P_h u - u) + (\alpha u + z - \mathrm{div}q, P_h u - u) \\
&\quad + (z - z_h(u), P_h u - u) + (\mathrm{div}(q - q_h(u)), P_h u - u) \\
&\quad + (z_h(u) - z_h, P_h u - u) + (\mathrm{div}(q_h(u) - q_h), P_h u - u) \\
&\quad + (z - z_h(u), u - u_h) + (\mathrm{div}(q - q_h(u)), u - u_h) \\
&\quad + (z_h(u) - z_h, u - u_h) + (\mathrm{div}(q_h(u) - q_h), u - u_h). \quad (4.16)
\end{aligned}
$$

First, we find the bounds of the last two terms on the right-hand side of (4.16). Taking $v = z_h - z_h(u)$ in (4.1) and $\tau = q_h - q_h(u)$ in (4.2), and setting $v = y_h - y_h(u)$ in (4.3) and $\tau = p_h - p_h(u)$ in (4.4), we have

$$(z_h(u) - z_h, u - u_h) + (\mathrm{div}(q_h(u) - q_h), u - u_h) = -\|y_h - y_h(u)\|^2_{L^2(\Omega)}$$
$$- \|p_h - p_h(u)\|^2_{L^2(\Omega)} \leq 0. \quad (4.17)$$

Moreover, if $u \in H^2(\Omega)$, $z \in H^2(\Omega)$ and q in $(H^2(\Omega))^2$, we have

$$
\begin{aligned}
(\alpha u + z - \mathrm{div}q, P_h u - u) &= ((\alpha u + z - \mathrm{div}q) - P_h(\alpha u + z - \mathrm{div}q), P_h u - u) \\
&\leq \|(\alpha u + z - \mathrm{div}q) - P_h(\alpha u + z - \mathrm{div}q)\|_{L^2(\Omega)} \\
&\quad \|P_h u - u\|_{L^2(\Omega)} \\
&\leq Ch^{k+1}(\|q\|^2_{H^2(\Omega)} + \|z\|^2_{H^2(\Omega)} + \|u\|^2_{H^2(\Omega)}). \quad (4.18)
\end{aligned}
$$

Therefore, by (4.16)-(4.18), projection operator properties and Cauchy-Schwartz inequality, we obtain

$$
\begin{aligned}
\alpha\|u - u_h\|_{L(\Omega)} &\leq Ch^{k+1}(\|q\|^2_{H^2(\Omega)} + \|z\|^2_{H^2(\Omega)} + \|u\|^2_{H^2(\Omega)}) + C(\|z - z_h(u)\|^2 \\
&\quad +\|\mathrm{div}(q - q_h(u))\|^2 + \|z_h(u) - z_h\|^2 + \|\mathrm{div}(q_h(u) - q_h)\|^2 \\
&\quad +\|u - u_h\|^2). \\
&\leq Ch^{k+1}(\|q\|^2_{H^2(\Omega)} + \|z\|^2_{H^2(\Omega)} + \|u\|^2_{H^2(\Omega)}) \\
&\quad +C(\|z - z_h(u)\|^2 + \|\mathrm{div}(q - q_h(u))\|^2 + \|u - u_h\|^2).
\end{aligned}
$$
$$(4.19)$$

It follows easily, from Lemma (3.2) and Lemma (4.1), that

$$
\|u - u_h\|_{L^2(\Omega)} \leq Ch^{k+1}M. \quad (4.20)
$$

Moreover it follows easily from Lemma (3.2), Lemma (4.1) and (4.20), that

$$
\|y - y_h\|_{L^2(\Omega)} + \|p - p_h\|_{H(\mathrm{div};\Omega)} + \|z - z_h\|_{L^2(\Omega)} + \|q - q_h\|_{H(\mathrm{div};\Omega)} \leq Ch^{k+1}M.
$$

Thus the theorem is proved. $\square$

5. Numerical Example

We illustrate the theoretical results of the previous section with a numerical example to solve the splitting positive definite mixed finite element methods for optimal control problems to approximate the state and control variables, by piecewise linear continuous function for whole domain of order $k = 1$.

To solve the optimal control problem numerically, we use the following preconditioning projection gradient method for n^{th} iteration. We now briefly describe the computational processes to be used for solving the numerical examples in this section. For the minimization of our problem, we define the

iterative scheme $(n = 0, 1, 2, \cdots)$

$$
\begin{aligned}
(u_{n+\frac{1}{2}}, v_h) &= (u_n, v_h) - \rho_n(\alpha u_n + z_n - \mathrm{div}q_n, v_h), \\
(y_n, \mathrm{v}_h) + (\mathrm{div}p_n, \mathrm{v}_h) &= (f, \mathrm{v}_h) + (u, \mathrm{v}_h), \\
(\mathrm{div}p_n, \mathrm{div}\tau_h) &= (f, \mathrm{div}\tau_h) + (u_n, \mathrm{div}\tau_h), \\
(A^{-1}z_n, \mathrm{v}_h) &= -(y_n - y_d, \mathrm{v}_h), \\
(\mathrm{div}q_n, \mathrm{div}\tau_h) &= -(p_n - p_d, \tau_h) - (\mathrm{div}\tau_n, z_h), \\
u_{n+1} &= P_K(u_{k+\frac{1}{2}}),
\end{aligned}
$$

where $\rho_n = 1$. Now define the projection operator $P_K : \mathrm{W} \to \mathrm{K}$ satisfying the following bilinear form. For given $\mathrm{w} \in \mathrm{W}$ find $P_K w \in \mathrm{K}$ such that

$$
(P_K w - w, P_K w - w) = \min_{u \in K}(u - w, u - w),
$$

which is equivalent to

$$
(P_K w - w, v - P_K w) \geq 0 \ \forall v \in K.
$$

Let W_n be the coordinates of w_n in $\mathbb{R}^n$, and $DJ(u_n)$ be the gradient of $J_{(u_n)}$, then the matrix from of the above problem reads as

$$
U_{n+\frac{1}{2}} = U_n - \rho_n M_h^{-1} DJ(u_n),
$$

where M_h is the mass matrix of $(\cdot, \cdot)_K$, the preconditioner.

Example 5.1. Our numerical example is the following optimal control problem for splitting positive definite mixed finite element space:

$$
\begin{aligned}
\underset{u \in K \subset L^2(\Omega)}{\text{minimize}} &\left\{ \frac{1}{2}\left(\|p - p_d\|^2 + \|y - y_d\|^2 + \alpha\|u\|^2 \right) \right\}, \\
(y, \mathrm{v}) + (\mathrm{div}p, \mathrm{v}) &= (f, \mathrm{v}) + (u, \mathrm{v}), \forall \mathrm{v} \in V, \\
(\mathrm{div}p, \mathrm{div}\tau) &= (f, \mathrm{div}\tau) + (u, \mathrm{div}\tau), \forall \tau \in W,
\end{aligned}
$$

and the admissible set is $K = \left\{ v \in L^2(\Omega) : \int_\Omega v \geq 0 \right\}$. We take $\Omega = [0,1] \times [0,1]$. Then A is the identity matrix and $f = 0$, and the problem becomes

$$
\begin{aligned}
y(x) &= \sin(\pi x_1)\sin(\pi x_2), \quad p(x) = -A(x)\nabla y, \\
q(x) &= (\sin(\pi x_1), \sin(\pi x_2)), \quad z(x) = -\pi^2 \sin(\pi x_1)\sin(\pi x_2).
\end{aligned}
$$

The source function f and the desired states y_d and p_d can be determined using the above functions. Now we easily obtain $\int_\Omega z = -4$. From the projection operator, we have

$$u = \max(0, \bar{z}) - z + \operatorname{div}q.$$

Thus we have $\int_\Omega u = 4 \geq 0$. Convergence plots are shown in Figure 1. Table 1 disposals the error estimate data of $L^2(\Omega)$-norm splitting positive definite mixed finite element space are approximated the state and control variables respectively.

h	$\|u - u_h\|$	Rate	$\|y - y_h\|$	Rate	$\|p - p_h\|$	Rate	$\|q - q_h\|$
$\frac{1}{16}$	3.012e-002	—	6.967e-003	—	3.561e-003	—	3.215e-003
$\frac{1}{32}$	8.132e-003	1.89	1.818e-003	1.93	8.912e-004	2.0	8.625e-004
$\frac{1}{64}$	2.150e-003	1.91	4.602e-004	1.98	2.382e-004	1.90	2.236e-004
$\frac{1}{128}$	5.435e-004	1.98	1.154e-004	1.99	5.817e-005	2.03	5.431e-005

Table 1: Numerical results for control and state solutions.

6. Conclusion

We have developed a priori error analysis of splitting positive definite mixed finite element method for optimal control problems governed by elliptic partial differential equations. Our resulting procedures for the control constraint are splitting positive symmetric definite matrix for whole domain by mixed finite element methods seems to be new. We have used piecewise linear polynomial functions to approximate the control variable for RT space.
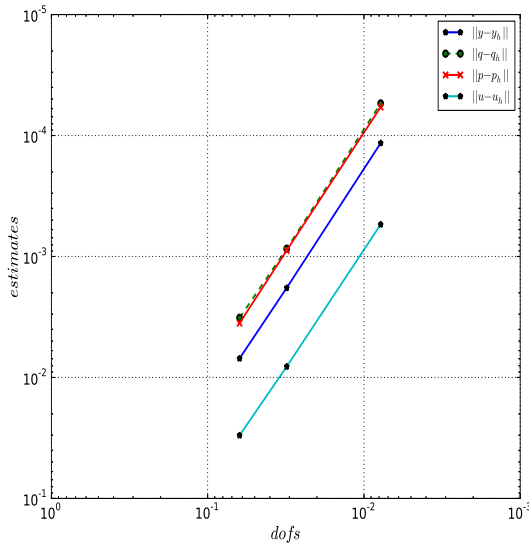
Acknowledgments

Figure 1: Convergence plot for control and state variables.

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