International Journal of Applied Mathematics

Volume 28 No. 3 2015, 307-317
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v28i3.9

TWO DIMENSIONAL STEADY STATE THERMOELASTIC PROBLEM OF RECTANGULAR SLAB WITH UNIFORM HEAT GENERATION SUBJECTED TO SYMMETRIC HEAT SUPPLY

A.S. Sewalkar[1 §], S.A. Meshram[2]
[1,2]Department of Mathematics
P.G.T.D. RTM Nagpur University, Nagpur, INDIA

Abstract: This paper discusses the two dimensional steady state thermoelastic problem of rectangular slab with internal heat generation q'''/k of the cross section $(2L \times 2l)$ with the sides of the slab exposed to symmetric heat supply. The heat transfer coefficient is considered to be large. The temperature of the problem is obtained by applying variable separation method Airy's stress function method is used to analyze the thermal displacements and stresses and results are discussed graphically.

AMS Subject Classification: 35A25, 74M99, 74K20
Key Words: thermoelastic problem, rectangular slab, two dimensions, internal heat generation, large heat transfer coefficient

1. Introduction

The study of thermoelastic problem in rectangular slab has many applications in aeronautical engineerings, naval and structural engineerings, automobile engineering depending upon the thickness of the slab. The previous related studies are used to formulate and develop the analysis of the problem. Y. Tanigawa

Received: April 23, 2015
§Correspondence author

and Y. Komatsubara [7] have studied the thermal stress analysis of rectangular plate and its stress intensity factor for compressive stress field. Vihak et al [8] have discussed the solution of the plane thermoelasticity problem for a rectangular domain. R.J. Adams and C.W. Best [1] have investigated thermoelastic vibrations of a laminated rectangular plate subjected to thermal shock. Y. Ootao et. al. [4] have studied the theoretical analysis of a three dimensional transient thermal stress problem for non-homogeneous hollow circular cylinder due to moving heat source in axial direction from inner and outer surface. W.K. Dange et. al. [2] have analysed three dimensional inverse transient thermoelastic problem of thin rectangular plate. K.R. Gaikwad and K.P. Ghadle [3] have studied three dimensional non homogeneous thermoelastic problem in three dimensional thick rectangular plate due to internal heat generation. Owing to the previous studies we have discussed the thermoelastic problem of rectangular slab in two dimensions (x,z) with internal heat generation.

2. Statement of the Heat Conduction Problem

Consider a steady state thermoelastic problem of heat generation q''' in a two dimensional rectangular slab of cross section $2L \times 2l$. The sides of the slab are subjected to symmetric heat supply. The heat transfer coefficient is large. The steady state two dimensional heat conduction equation in Cartesian coordinates is given as:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q'''}{k} = 0 \tag{1}$$

with the boundary conditions

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \qquad \text{for} \quad 0 \le z \le l, \tag{2}$$

$$T|_{x=L} = T_\infty \qquad \text{for} \quad 0 \le z \le l, \tag{3}$$

$$\left.\frac{\partial T}{\partial x}\right|_{z=0} = 0 \qquad \text{for} \quad 0 \le x \le L, \tag{4}$$

$$T|_{z=l} = T_\infty \qquad \text{for } 0 \le x \le L, \tag{5}$$

where T is temperature, k is thermal conductivity and q''' is uniform heat generation.

Airy's stress function method is used for the analytical development of the thermoelastic field. The equation is given by the relation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)^2 U=-\alpha E\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right) T, \tag{6}$$

where α and E are the linear coefficients of the thermal expansion and young's modulus of elasticity of the materials of the rectangular plate, respectively.

The displacement components U_x and U_z in the x and z direction represented in the integral form and the stress components in terms of U are given by

$$U_x=\int\left(\frac{1}{E}\left(\frac{\partial^2 U}{\partial z^2}-\nu\frac{\partial^2 U}{\partial x^2}\right)+\alpha T\right)dx \tag{7}$$

$$U_z=\int\left(\frac{1}{E}\left(\frac{\partial^2 U}{\partial x^2}-\nu\frac{\partial^2 U}{\partial z^2}\right)+\alpha T\right)dz \tag{8}$$

$$\sigma_{xx}=\frac{\partial^2 U}{\partial z^2} \tag{9}$$

$$\sigma_{zz}=\frac{\partial^2 U}{\partial x^2} \tag{10}$$

$$\sigma_{xz}=\frac{-\partial^2 U}{\partial x\partial z}, \tag{11}$$

where ν is the poisson's ratio of the material of slab.

3. Determination of Temperature

Since the equation (1) is non-homogeneous, it is not separable. The non-homogeneity arises from the q'''/k term.

Now before proceeding the solution, we make boundary conditions at $x=0$, $x=L$, $z=0$ and $z=l$ homogeneous by using $\theta=T\text{-}T_\infty$ which transforms equation (1) and (2) to (5) as follows:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial z^2}+\frac{q'''}{k}=0 \tag{12}$$

with boundary conditions

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0 \qquad \text{for} \quad 0\le z\le l, \tag{13}$$

$$\theta|_{x=L} = 0 \qquad \text{for} \quad 0 \le z \le l, \tag{14}$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{z=0} = 0 \qquad \text{for} \quad 0 \le x \le L, \tag{15}$$

$$\theta|_{z=l} = 0 \qquad \text{for} \quad 0 \le x \le L. \tag{16}$$

The solution of the problem is now assumed to be

$$\theta\left(x, z\right) = \psi\left(x, z\right) + \phi\left(x\right), \tag{17}$$

$$\theta\left(x, z\right) = \psi\left(x, z\right) + \phi\left(z\right), \tag{18}$$

where ψ and ϕ are arbitrary functions.

By including the heat generation term q''' in one dimension term $\phi(x)$ or $\phi(z)$, $\psi(x,\ z)$ can be made homogeneous.

By using the variable separation method, we obtain temperature function $T(x, z)$ as

$$T\left(x, z\right) = \frac{-2q'''L^2}{k}\sum_{n=1}^{\infty}\frac{(-1)^n}{(\lambda_n L)^3}\left[\frac{\cosh\lambda_n z}{\cosh\lambda_n l}\right]\cos\lambda_n x + \frac{1}{2}\frac{q'''}{k}L^2\left(1 - \left(\frac{x}{L}\right)^2\right), \tag{19}$$

where $\lambda_n = \dfrac{(2n+1)\,\frac{\pi}{2}}{L}$, $n =$ 0,1,2,3......

4. Determination of Airy's Stress Function

Substituting the value of temperature function (19) in (6), we obtain Airy's stress function U as

$$\begin{aligned} U\left(x, z\right) = -\alpha E &\left[\tfrac{-2q'''L^2}{K}\textstyle\sum_{n=1}^{\infty}\tfrac{(-1)^n}{(\lambda_n L)^3}\left[\tfrac{\cosh\lambda_n z}{\cosh\lambda_n l}\right]\cos\lambda_n x\right. \\ &\left.+\tfrac{1}{2}\tfrac{q'''}{k}L^2\left[1 - \left(\tfrac{x}{L}\right)^2\right]\right]. \end{aligned} \tag{20}$$

5. Determination of Displacements

Substituting the value of T and U from equations (19) and (20) in (7) and (8), we obtain displacements as

$$\begin{aligned}U_x = -\int \Big\{ & \tfrac{-2\alpha q''' L^2}{K} \Big[(1+\nu)\textstyle\sum_{n=1}^{\infty} \tfrac{(-1)^n}{(\lambda_n L)^3} \Big[\tfrac{\cosh \lambda_n z}{\cosh \lambda_n l}\Big] \lambda_n^2 \cos \lambda_n x \\ & + \textstyle\sum_{n=1}^{\infty} \tfrac{(-1)^n}{(\lambda_n L)^3} \Big[\tfrac{\cosh \lambda_n z}{\cosh \lambda_n l}\Big] \cos \lambda_n x \\ & + \tfrac{2\nu q'''}{k} \Big[1 + \tfrac{L^2}{2}\Big(1 - \big(\tfrac{x}{L}\big)^2\Big)\Big]\Big\} \, dx, \end{aligned} \tag{21}$$

$$\begin{aligned}U_z = -\int \alpha \Big\{ & \tfrac{2q''' L^2}{K} (1+\nu) \textstyle\sum_{n=1}^{\infty} \tfrac{(-1)^n}{(\lambda_n L)^3} \Big[\tfrac{\cosh \lambda_n z}{\cosh \lambda_n l}\Big] \lambda_n^2 \cos \lambda_n x \\ & - \tfrac{2q''' L^2}{K} \textstyle\sum_{n=1}^{\infty} \tfrac{(-1)^n}{(\lambda_n L)^3} \Big[\tfrac{\cosh \lambda_n z}{\cosh \lambda_n l}\Big] \cos \lambda_n x \\ & + \tfrac{q'''}{k} \Big[\tfrac{L^2}{2}\Big(1 - \big(\tfrac{x}{L}\big)^2\Big) - 1\Big]\Big\} \, dz. \end{aligned} \tag{22}$$

6. Determination of Stress Functions

Substituting the values of U from (20) in equations (9), (10) and (11), we obtain the stresses as

$$\sigma_{xx} = -\alpha E \left[\frac{-2q''' L^2}{K} \sum_{n=1}^{\infty} \frac{(-1)^n}{(\lambda_n L)^3} \left[\frac{\cosh \lambda_n z}{\cosh \lambda_n l}\right] \lambda_n^2 \cos \lambda_n x \right], \tag{23}$$

$$\sigma_{zz} = -\alpha E \left[\frac{2q''' L^2}{K} \sum_{n=1}^{\infty} \frac{(-1)^n}{(\lambda_n L)^3} \left[\frac{\cosh \lambda_n z}{\cosh \lambda_n l}\right] \lambda_n^2 \cos \lambda_n x - \frac{q'''}{K} \right], \tag{24}$$

$$\sigma_{xz} = -\alpha E \left[\frac{-2q''' L^2}{K} \sum_{n=1}^{\infty} \frac{(-1)^n}{(\lambda_n L)^3} \left[\frac{\sinh \lambda_n z}{\cosh \lambda_n l}\right] \lambda_n^2 \sin \lambda_n x + \frac{q'''}{K} x \right]. \tag{25}$$

7. Numerical Results

The numerical calculations have been carried out for a steel (SN50C) plate with the properties $K = 15.9 \times 10^{-6}$ (m^2/s), $\nu = 0281, \lambda = 590[\mathrm{Wm}_1 K_1]$.

We set

$$L = 0............1.5$$

$$l = 0............2.5$$

and take

$$\lambda_n = \frac{(2n+1)\frac{\pi}{2}}{L}, \quad n = 0, 1, 2, 3.......$$

$$x = 0............1.5$$

$$z = 0............2.5$$

$$q''' = 100\ Btu/ft/hr.$$

8. Discussion

Figures 1 to 2 show graphs for temperature. Temperature shows variation in x direction. Temperature shows increasing values near x=0 steadily decreases for increasing values of x. While in z direction it shows constant value for increasing values of z. This is caused due to steady state condition and sides are subjected to heat distribution. Also the points x=0 and near x=0 get more heat.

Figures 3 to 6 show graphs for displacement U_x and U_z in x & z direction respectively. The displacement U_x in x direction increases near x=0 and shows steady values for increasing values of x and the displacement U_x increases in z-direction for different values of x. The displacement U_z shows constant value for increasing values of x and z.

The stresses σ_{xx} in Figure 7 in x direction show steady value for increasing value of x while σ_{xx} in Figure 8 in z-direction initially shows steady values but for values $z \geq 1$ it shows increasing and decreasing pattern. This is caused because the displacement U_x increases in z-direction.

The stresses σ_{zz} in Figures 9 to 10 in both x and z directions show steady value for increasing values of x and z because of displacement trends. U_z in both directions show steady values.

The stresses σ_{xz} in Figure 11 increase for increasing values of x while σ_{xz} in Figure 12 in z direction shows steady pattern with increasing values.

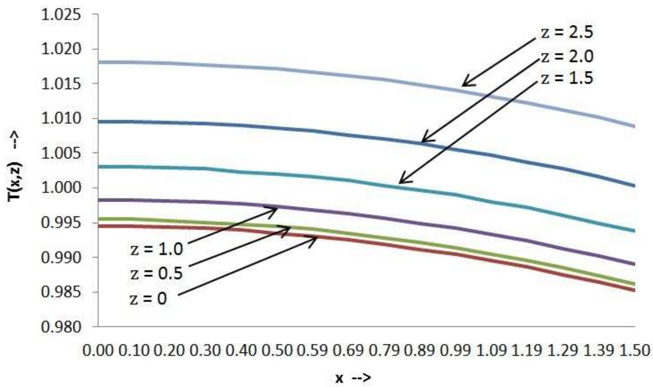

Figure 1: $T(x,z)$ versus x for different values of z

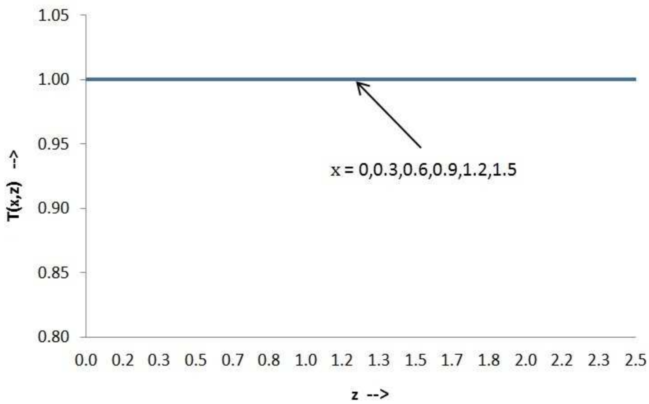

Figure 2: $T(x,z)$ versus z for different values of x

Conclusions

From these observations we can conclude that most affected direction is the x-direction. In z-direction we observe less variation because the temperature gradient is less in z direction than in x direction.

References

[1] R.J. Adams and C.W. Best, Thermoelastic vibrations of a laminated rect-

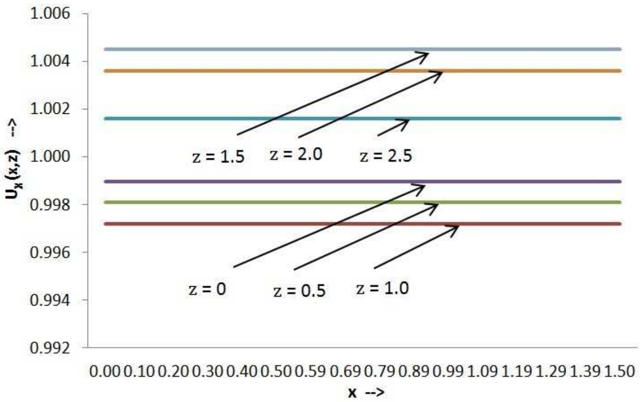

Figure 3: $U_x\ (x, z)$ versus x for different values of z

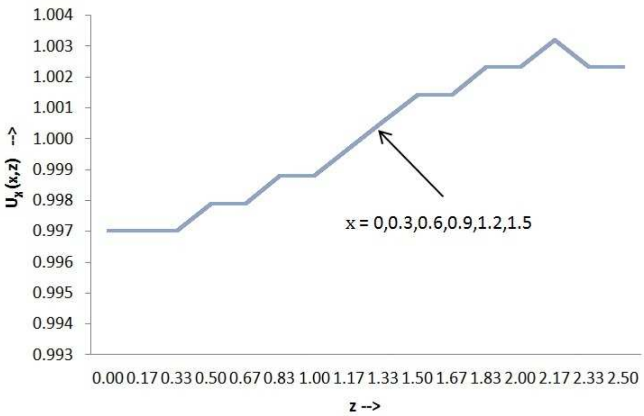

Figure 4: $U_x\ (x, z)$ versus z for different values of x

angular plate subjected to thermal shock, *J. of Thermal Stresses*, **22** (1999), 875-895.

[2] W.K. Dange, N. W. Khobragade and M.H. Durge, Three dimensional inverse transient thermoelastic problem of a thin rectangular plate, *Int. J. of Appl. Maths.*, **23** (2010), 207-222.

[3] K.R. Gaikwad and K.P. Ghadle, Three dimensional non homogeneous ther-

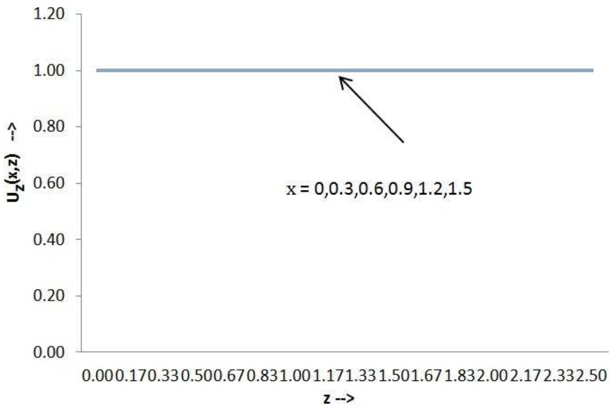

Figure 5: $U_z\ (x, z)$ versus x for different values of z

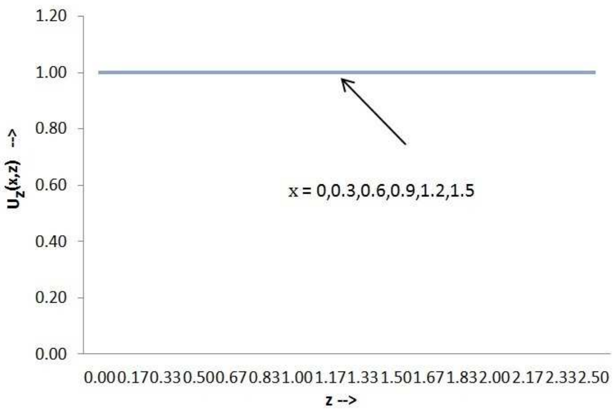

Figure 6: $U_z\ (x, z)$ versus z for different values of x

moelastic problem in a thick rectangular plat due to internal heat generation, *Southern Africa J. of Pure and Applied Mathematics*, **5** (2011), 26-38.

[4] Y. Ootao, T. Akai and Y. Tanigawa, Three dimensional transient thermal stress analysis of a non-homogeneous hollow circular cylinder due to a moving heat source in the axial direction, *J. of Thermal Stresses*, **18** (1995), 497-512.

[5] N.M. Ozisik, *Boundary Value Problem of Heat Conduction*, International

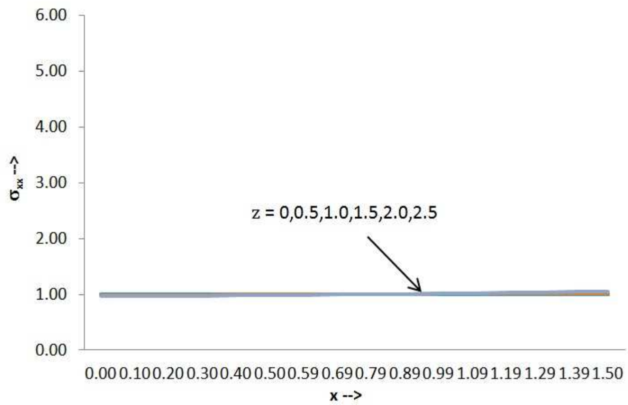

Figure 7: $\sigma_{xx}\ (x, z)$ versus x for different values of z

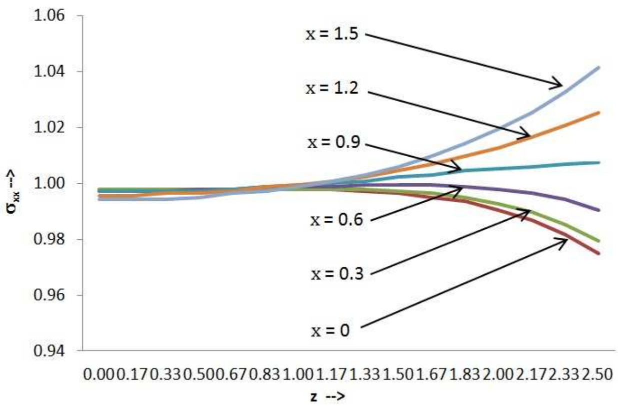

Figure 8: $\sigma_{xx}\ (x, z)$ versus z for different values of x

Textbook Company, Seranton, Pennsylvania, 84-101 (1968).

[6] N. Noda, R. B. Hetnarski and Y. Tanigawa, *Thermal Stresses*, Second Ed., Taylor & Fransis, New York (2003).

[7] Y. Tanigawa and Y. Komatsubara, Thermal stress analysis of a rectangular plate and its thermal stress intensity factor for compressive stress field, *J. of Thermal Stresses*, **20** (1997), 517-542.

[8] V.M. Vihak, M.Y. Yuzvyak, and A.V. Yasinskij, The solution of the plane

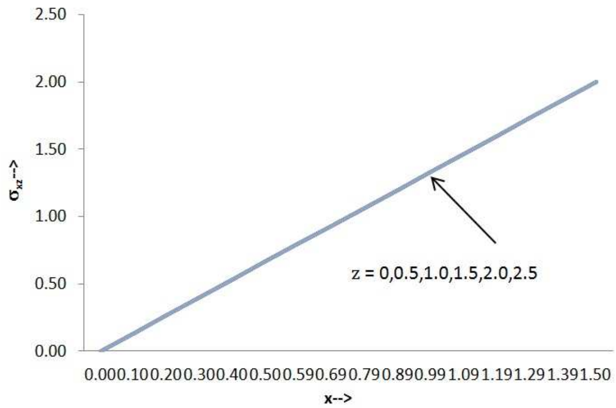

Figure 9: $\sigma_{xz}\ (x, z)$ versus x for different values of z

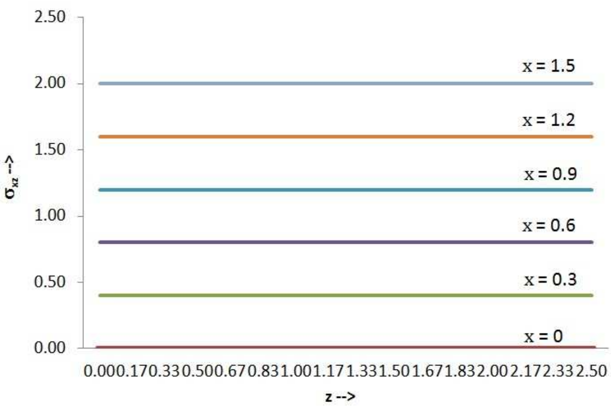

Figure 10: $\sigma_{xz}\ (x, z)$ versus z for different values of x

thermoelasticity problem for a rectangular domain, *J. of Thermal Stresses*, **21** (1998), 545-561.

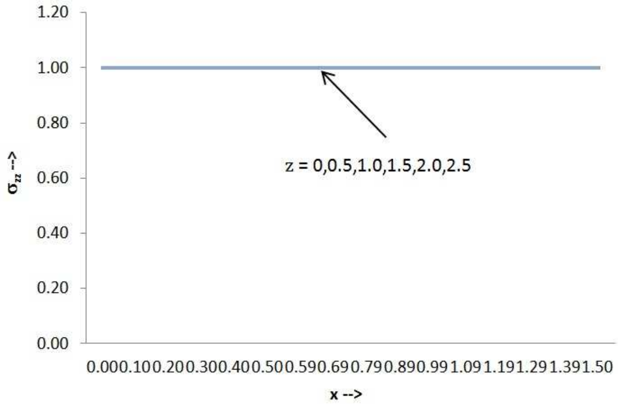

Figure 11: $\sigma_{zz}\ (x, z)$ versus x for different values of z

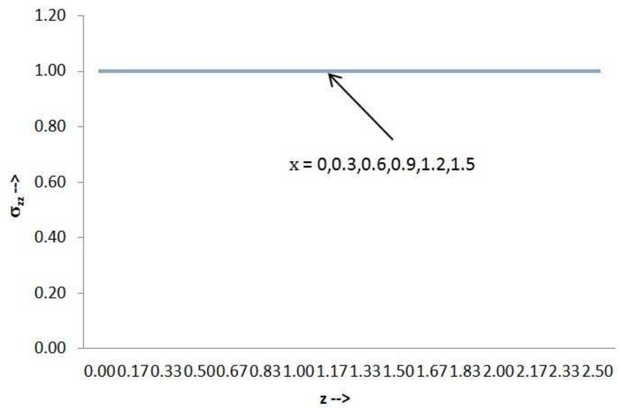

Figure 12: $\sigma_{zz}\ (x, z)$ versus z for different values of x