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# HEAT, RESOLVENT AND WAVE KERNELS WITH TRIPLE INVERSE SQUARE POTENTIAL ON THE EUCLIDIAN SPACE $(R^+)^3$

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**Abstract:** In this paper the heat, resolvent and wave kernels associated to the Schrödinger operator with triple-inverse square potential on the Euclidian space $(R^+)^3$ are given in explicit forms.

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**Key Words:** heat kernel, wave kernel, resolvent kernel, triple-inverse square potential, Lauricella hypergeometric function

## 1. Introduction

In this paper we give explicit formulas for the Schwartz integral kernels of the heat, resolvent and wave operators $e^{t\Delta_{\nu,\mu,\eta}}$, $\left(\Delta_{\nu,\mu,\eta} + \lambda^2\right)^{-1}$ and $\cos t\sqrt{-\Delta_{\nu,\mu,\eta}}$ attached to the Schrödinger operator with triple-inverse square potential on the Euclidian space $(R^+)^3$:

$$\Delta_{\nu,\mu,\eta} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{1/4 - \nu^2}{x^2} + \frac{1/4 - \mu^2}{y^2} + \frac{1/4 - \eta^2}{z^2}, \tag{1.1}$$

where $\nu, \mu, \eta$ are real parameters.

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The inverse square potential is an interesting potential which arises in several contexts, one of them being the Schrödinger equation in non relativistic quantum mechanics. For example the Hamiltonian for a spin zero particle in Coulomb field gives rise to a Schrödinger operator involving the inverse square potential see Case [5]. Note that the Schrödinger operator with bi-inverse square potential in the Euclidian plane is considered in Boyer [2] and Ould Moustapha [10].

First of all we recall the following formulas for the modified Bessel function of the first kind $I_\nu$ and the Hankel function of the first kind $H_\nu^{(1)}$:

$$I_\nu(z) = \frac{(2z)^\nu e^z}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_0^1 e^{-2zt} \left[t(1-t)\right]^{\nu-1/2} dt \tag{1.2}$$

(see Temme [12], p. 237),

$$I_\nu(x) \sim \frac{(x/2)^\nu}{\Gamma(\nu+1)} \qquad x \longrightarrow 0 \quad \nu \neq -1, -2, ...., \tag{1.3}$$

(see Temme [12], p. 234),

$$I_\nu(x) \sim e^x (2\pi x)^{-1/2} \qquad x \longrightarrow \infty \tag{1.4}$$

(see Temme [12], p. 240),

$$H_\nu^{(1)}(z) = \frac{2}{i\sqrt{\pi}\Gamma(1/2-\nu)} (z/2)^{-\nu} \int_1^\infty e^{izt} \left(t^2-1\right)^{-\nu-1/2} dt \tag{1.5}$$

(see Erdélyi et al. [7], p. 83),

$$H_\nu^{(1)}(z\sqrt{\alpha^2}) = \frac{-i}{\pi} e^{-i\nu\pi/2} (\alpha^2)^{\nu/2} \int_0^\infty e^{i\frac{z}{2}\left(t+\frac{\alpha^2}{t}\right)} t^{-\nu-1} dt, \tag{1.6}$$

$\mathcal{I}\, z > 0$ and $\mathcal{I}\, \alpha^2 z > 0$ (see Magnus et al. [9], p. 84). Recall also the three variables Lauricella series $F_A^{(3)}$ hypergeometric function ($|z| + |z'| + |z''| < 1$) (see Appell et al. [1], p. 114):

$$F_A^{(3)}(\alpha, \beta, \beta', \beta'', \gamma, \gamma', \gamma'', z, z', z'')$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha)_{m+n+p} (\beta)_m (\beta')_n (\beta'')_p}{(\gamma)_m (\gamma')_n (\gamma'')_p m! n! p!} z^m z'^n z''^p \tag{1.7}$$

and its integral representation (see Appell et al. [1], p.115) for $\Re\,\beta > 0$, $\Re\,\beta' > 0$, $\Re\,(\gamma-\beta) > 0$, $\Re\,(\gamma'-\beta') > 0$ and $\Re\,(\gamma''-\beta'') > 0$:

$$F_A^{(3)}(\alpha,\beta,\beta',\beta'',\gamma,\gamma',\gamma'',z,z',z'')$$

$$= c\int_0^1\int_0^1\int_0^1 (1-u)^{\gamma-\beta-1}(1-v)^{\gamma'-\beta'-1}(1-w)^{\gamma''-\beta''-1}$$
$$\times u^{\beta-1}v^{\beta'-1}w^{\beta''-1}\left(1-uz-vz'-wz''\right)^{-\alpha}dudvdw, \tag{1.8}$$

where

$$c = \frac{\Gamma(\gamma)\Gamma(\gamma')\Gamma(\gamma'')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\beta'')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')\Gamma(\gamma''-\beta'')}. \tag{1.9}$$

Recall also the following formulas for the heat kernel associated to the Schrődinger operator with inverse square potential $L_\nu = \frac{\partial^2}{\partial x^2} + \frac{1/4-\nu^2}{x^2}$ (see Calin et al. [4], p. 68):

$$e^{tL_\nu} = \frac{(xx')^{1/2}}{2t}e^{\frac{-(x^2+x'^2)}{4t}}I_\nu(\frac{xx'}{2t}), \tag{1.10}$$

where $I_\nu$ is the modified Bessel function of the first kind.

**Proposition 1.1.** *The Schwartz integral kernel of the heat operator with triple-inverse square potential $e^{t\Delta_{\nu,\mu,\eta}}$ can be written for $p=(x,y,z)$, $p'=(x',y',z')\in(R^+)^3$ and $t\in R_+$ as*
$$H_{\nu,\mu,\eta}(t,p,p') = \frac{(xx'yy'zz')^{1/2}}{8t^3}e^{-(|p|^2+|p'|^2)/4t}$$

$$\times I_\nu(xx'/2t)I_\mu(yy'/2t)I_\eta(zz'/2t), \tag{1.11}$$

*where $I_\nu$ is the modified Bessel function of the first kind.*

*Proof.* The formula (1.11) is a direct consequence of the formula (1.10) and the properties of the operator (1.1).

## 2. Resolvent Kernel with Triple-Inverse Square Potential on the Euclidian Space $(R^+)^3$

**Theorem 2.1.** *The Schwartz integral kernel for the resolvent operator $\left(\Delta_{\nu,\mu,\eta}+\lambda^2\right)^{-1}$ is given by the formula*

$$G_{\nu,\mu}(\lambda,p,p') = c_1(xx')^{\nu+1/2}(yy')^{\mu+1/2}(zz')^{\eta+1/2}$$

$$\int_0^1\int_0^1\int_0^1\left(\frac{\lambda}{\sqrt{|p-p'|^2+4xx'u+4yy'v+4zz'w}}\right)^{\nu+\mu+\eta+2}$$
$$\times H^{(1)}_{\nu+\mu+\eta+2}(\lambda\sqrt{|p-p'|^2+4xx'u+4yy'v+4zz'w})$$
$$\times [u(1-u)]^{\nu-1/2}[v(1-v)]^{\mu-1/2}[w(1-w)]^{\eta-1/2}dudvdw, \tag{2.1}$$

*where $H^{(1)}_\nu$ is the Hankel function of the first kind and*

$$c_1=-\frac{i\,2^{\nu+\mu+\eta-1}}{8\sqrt{\pi}\Gamma(\nu+1/2)\Gamma(\mu+1/2)\Gamma(\eta+1/2)}.$$

*Proof.* We use the well known formula connecting the resolvent and the heat kernels:

$$G_{\nu,\mu,\eta}(\lambda,p,p')=\int_0^\infty e^{\lambda^2 t}H_{\nu,\mu,\eta}(t,p;p')dt;\qquad \mathcal{R}e\lambda^2<0. \tag{2.2}$$

we combine the formulas (2.2), (1.11) and (1.2) then use the formulas (1.3), (1.4) to apply the Fubini theorem and in view of the formula (1.6) we get the formula (2.1) and the proof of Theorem 2.1 is finished.

**Theorem 2.2.** *The Schwartz integral kernel of the resolvent operator $\left(\Delta_{\nu,\mu,\eta}+\lambda^2\right)^{-1}$ can be written as*

$$G_{\nu,\mu,\eta}(\lambda,p,p')=c_2(xx')^{\nu+1/2}(yy')^{\mu+1/2}(zz')^{\eta+1/2}$$
$$\times\int_{|p-p'|}^\infty e^{i\lambda s}\left(s^2-|p-p'|^2\right)^{-5/2-\nu-\mu-\eta}$$
$$\times F^{(3)}_A(\alpha,b_1,b_2,b_3,2b_1,2b_2,2b_3,$$
$$\frac{4xx'}{s^2-|p-p'|^2},\frac{-4yy'}{s^2-|p-p'|^2},\frac{-4zz'}{s^2-|p-p'|^2})ds \tag{2.3}$$

*with $\alpha=5/2+\nu+\mu+\eta$, $b_1=1/2+\nu$, $b_2=1/2+\mu$ and $b_3=1/2+\eta$,* $c_2=-\frac{2^{2\nu+2\mu+2\eta-1}}{\sqrt{\pi}}$

$$\times\frac{\Gamma(1/2+\nu)\Gamma(1/2+\mu)\Gamma(1/2+\eta)}{\Gamma(2\nu+1)\Gamma(2\mu+1)\Gamma(2\eta+1)\Gamma(-1/2-\nu-\mu-\eta)}, \tag{2.4}$$

*where $F^{(3)}_A$ is the three variables Lauricella hypergeometric function given in (1.7).*

*Proof.* We use formulas (2.1) and (1.5) as well as the Fubini theorem to obtain the announced formula (2.3).

## 3. Wave Kernel with Triple-Inverse Square Potential on the Euclidian Space $(R^+)^3$

It is known that the energy and information can only be transmitted with finite speed, smaller or equal to the speed of light. The mathematical framework, which allows an analysis and proof of this phenomenon, is the theory the wave equation. The result, which may be obtained, runs under the name finite propagation speed (see Cheeger et al. [6]). The following theorem illustrates this principle for the case of the the Schrödinger operator with triple-inverse square potential.

**Theorem 3.1.** (Finite propagation speed) *Let $W_{(\nu,\mu,\eta)}(t,p,p')$ be the Schwartz integral kernel of the wave operator $\frac{\sin t\sqrt{-\Delta_{\nu,\mu,\eta}}}{\sqrt{-\Delta_{\nu,\mu,\eta}}}$, then we have*

$$W_{(\nu,\mu,\eta)}(t,p,p') = 0 \quad \textit{whenever} \quad |p-p'| > t. \tag{3.1}$$

*Proof.* The proof of this result use an argument of analytic continuation from the identity

$$\frac{\sin t\sqrt{-\Delta_{\nu,\mu,\eta}}}{\sqrt{-\Delta_{\nu,\mu,\eta}}} = \frac{1}{2i}\left(\frac{e^{it\sqrt{-\Delta_{\nu,\mu,\eta}}}}{\sqrt{-\Delta_{\nu,\mu,\eta}}} - \frac{e^{-it\sqrt{-\Delta_{\nu,\mu,\eta}}}}{\sqrt{-\Delta_{\nu,\mu,\eta}}}\right). \tag{3.2}$$

We recall the formula from [13], p.50:

$$\frac{e^{-t\lambda}}{t} = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-ut^2}u^{-1/2}e^{-\lambda^2/4u}du. \tag{3.3}$$

By setting $t = \sqrt{-\Delta_{\nu,\mu,\eta}}$ and $\lambda = s$ in (3.3), we can write

$$e^{-s\sqrt{-\Delta_{\nu,\mu,\eta}}} = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-s^2/4u}u^{-1/2}e^{u\Delta_{\nu,\mu,\eta}}du \tag{3.4}$$

and let $P_{\nu,\mu,\eta}(s,p,p')$ be the integral kernel of $e^{-s\sqrt{-\Delta_{\nu,\mu,\eta}}}$. Then we can write

$$P_{\nu,\mu,\eta}(s,p,p') = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-s^2/4z}z^{-1/2}H_{\nu,\mu,\eta}(z,p,p')dz, \tag{3.5}$$

where $H_{\nu,\mu,\eta}(t,p,p')$ is the heat kernel with the triple-inverse square potential given by (1.11).

Consider the integral

$$J(\tau) = \frac{1}{\sqrt{\pi}} \int_0^\infty |e^{-\tau^2/4z} z^{-1/2} H_{\nu,\mu,\eta}(z,p,p')|dz. \tag{3.6}$$

Using (1.11) we have

$$J(\tau) = \frac{(xx'yy'zz')^{1/2}}{8\sqrt{\pi}} \int_0^\infty |e^{-\tau^2/4s} s^{-7/2} e^{-(|p|^2+|p'|^2)/4s}|$$
$$\times |I_\nu((xx')/2s) I_\mu((yy')/2s) I_\eta((zz')/2s)|ds. \tag{3.7}$$

From (3.2) we have

$$W_{\nu,\mu,\eta}(t,x,x') = \frac{1}{2i}\left(P_{\nu,\mu,\eta}(p,p',it) - P_{\nu,\mu,\eta}(p,p',-it)\right),$$

$$W_{\nu,\mu,\eta}(t,x,x') = \frac{1}{2i}\left(J(it) - J(-it)\right). \tag{3.8}$$

Now, set

$$J(\tau) = J_1(\tau) + J_2(\tau), \tag{3.9}$$

where

$$J_1(\tau) = \frac{(xx'yy'zz')^{1/2}}{8\sqrt{\pi}} \int_0^1 |e^{-\tau^2/4s} s^{-7/2} e^{-(|p|^2+|p'|^2)/4s}|$$
$$\times |I_\nu((xx')/2s) I_\mu((yy')/2s) I_\eta((zz')/2s)|ds, \tag{3.10}$$

$$J_2(\tau) = \frac{(xx'yy'zz')^{1/2}}{8\sqrt{\pi}} \int_1^\infty |e^{-\tau^2/4s} s^{-7/2} e^{-(|p|^2+|p'|^2)/4s}|$$
$$\times |I_\nu((xx')/2s) I_\mu((yy')/2s) I_\eta((zz')/2s)|ds. \tag{3.11}$$

Using formula (1.3), we see that the last integral $J_2(\tau)$ converges absolutely and it is analytic in $\tau$ for $\nu + \mu + \eta + 5/2 > 0$.

For the first integral $J_1(\tau)$ we obtain

$$J_1(\tau) = \frac{(xx'yy'zz')^{1/2}}{8\sqrt{\pi}} \int_1^\infty |e^{-\tau^2 s/4} s^{3/2} e^{-(|p|^2+|p'|^2)s/4}|$$
$$\times |I_\nu((xx')s/2) I_\mu((yy')s/2) I_\eta((zz')s/2)|ds, \tag{3.12}$$

and from the formula (1.4) we see that

$$J_1(\tau) \sim \frac{1}{8\pi^2} \int_1^\infty s^{3/2} e^{-\left(\tau^2+|p-p'|^2\right)\frac{s}{4}} ds \tag{3.13}$$

is analytic in $\tau$ and converges, if $\mathcal{R}e\left[\tau^2+|p-p'|^2\right] > 0$, hence the integral $J(\pm it)$ is absolutely convergent if $(\pm it)^2+|p-p'|^2>0$ (i.e. $|p-p'|>t$), and in view of (3.8) we have $W_{\nu,\mu,\eta}(t,p,p')=0$ for $|p-p'|>t$ and the proof of Theorem 3.1 is finished.

**Theorem 3.2.** *The Schwartz integral kernel for the wave operator* $\cos t\sqrt{-\Delta_{\nu,\mu,\eta}}$ *with bi-inverse square potential on the Euclidian plane can be written in the following two forms:*

$$w_{\nu,\mu,\eta}(t,p,p') = \frac{(xx'yy'zz')^{1/2}}{2i\sqrt{2\pi}t^6}\int_{-\infty}^{0+}\exp\left[-\frac{u}{2t^2}\left(|p|^2+|p'|^2-t^2\right)\right]$$
$$\times I_\nu\left(\frac{xx'}{t^2}u\right)I_\mu\left(\frac{yy'}{t^2}u\right)I_\eta\left(\frac{zz'}{t^2}u\right)u^{5/2}du \tag{3.14}$$

*and*

$$w_{\nu,\mu,\eta}(t,p,p') = 2\frac{(xx'yy'zz')^{1/2}}{\sqrt{2\pi}t^6}\int_0^{\infty}\exp\left[\frac{r}{2t^2}\left(|p|^2+|p'|^2-t^2\right)\right]$$
$$\times I_\nu\left(\frac{-xx'}{t^2}r\right)I_\mu\left(-\frac{yy'}{t^2}r\right)I_\eta\left(-\frac{zz'}{t^2}r\right)r^{5/2}dr, \tag{3.15}$$

*where $I_\nu$ is the first kind modified Bessel functions of order $\nu$.*

*Proof.* We start by recalling the formula (see Magnus et al [9], p.73):

$$\cos z = \sqrt{\pi z/2}J_{-1/2}(z), \tag{3.16}$$

where $J_\nu(.)$ is the Bessel function of first kind and of order $\nu$ defined by (see Magnus et al [9], p.83),

$$J_\nu(\alpha z) = \frac{z^\nu}{2i\pi}\int_{-\infty}^{0+} e^{(\alpha/2)(t-z^2/t)}t^{-\nu-1}dt, \tag{3.17}$$

provided that $\Re\,\alpha>0$ and $|argz|\le\pi$. Here we should note that the integral in (3.17) can be extended over a contour starting at $\infty$, going clockwise around 0, and returning back to $\infty$ without cutting the real negative semi-axis.

For $\nu=-1/2$ eq. (3.17) can be combined with eq. (3.16) to derive the following formula:

$$\cos\alpha z = \frac{1}{2i\sqrt{2\pi}}\int_{-\infty}^{0+} e^{(\alpha/2)(u-z^2/u)}u^{-1/2}du. \tag{3.18}$$

Putting $\alpha = 1$ and replacing the variable $z$ by the symbol $t\sqrt{-\Delta_{\nu,\mu}}$ in eq. (3.18), we obtain

$$\cos t\sqrt{-\Delta_{\nu,\mu}} = \frac{1}{2i\sqrt{2\pi}} \int_{-\infty}^{0+} e^{\left(u/2+(t^2/2u)\Delta_{\nu,\mu}\right)} u^{-1/2} du. \tag{3.19}$$

Finally making use of (1.11) in (3.19), we get after an appropriate change of variable, the formula (3.14).

To see the formula (3.15), set

$$J = \int_{-\infty}^{0+} \exp\left[-\frac{u}{2t^2}\left(|p|^2+|p'|^2-t^2\right)\right]$$
$$\times I_\nu\left(\frac{xx'}{t^2}u\right) I_\mu\left(\frac{yy'}{t^2}u\right) I_\eta\left(\frac{zz'}{t^2}u\right) u^{5/2} du \tag{3.20}$$

and

$$I = \int_0^{\infty} \exp\left[\frac{u}{2t^2}\left(|p|^2+|p'|^2-t^2\right)\right]$$
$$\times I_\nu\left(-\frac{xx'}{t^2}u\right) I_\mu\left(-\frac{yy'}{t^2}u\right) I_\eta\left(-\frac{zz'}{t^2}u\right) u^{5/2} du. \tag{3.21}$$

We have

$$J = J_1 + J_2 + J_3, \quad \text{where} \tag{3.22}$$

$$J_1 = \int_{\gamma_1} \exp\left[-\frac{u}{2t^2}\left(|p|^2+|p'|^2-t^2\right)\right]$$
$$\times I_\nu\left(\frac{xx'}{t^2}u\right) I_\mu\left(\frac{yy'}{t^2}u\right) I_\eta\left(\frac{zz'}{t^2}u\right) u^{5/2} du, \tag{3.23}$$

$$J_2 = \int_{\gamma_2} \exp\left[-\frac{u}{2t^2}\left(|p|^2+|p'|^2-t^2\right)\right]$$
$$\times I_\nu\left(\frac{xx'}{t^2}u\right) I_\mu\left(\frac{yy'}{t^2}u\right) I_\eta\left(\frac{zz'}{t^2}u\right) u^{5/2} du, \tag{3.24}$$

and

$$J_3 = \int_{\gamma_3} \exp\left[-\frac{u}{2t^2}\left(|p|^2+|p'|^2-t^2\right)\right]$$
$$\times I_\nu\left(\frac{xx'}{t^2}u\right) I_\mu\left(\frac{yy'}{t^2}u\right) I_\eta\left(\frac{zz'}{t^2}u\right) u^{5/2} du, \tag{3.25}$$

where the paths $\gamma_1$, $\gamma_2$ and $\gamma_3$ are given by

$$\gamma_1 : z = re^{i\pi}; \epsilon \le r < \infty \quad \text{(above the cut)}$$

$$\gamma_2 : z = re^{-i\pi}; \infty > r \geq \epsilon \quad \text{(below the cut)}$$

$$\gamma_3 : z = \epsilon e^{i\theta}; -\pi < \theta < \pi \quad \text{(rund the small circle)}$$

as $\epsilon \longrightarrow 0$, we have $J_1 \longrightarrow e^{7i\pi/2} I$, $J_2 \longrightarrow -e^{-7i\pi/2} I$ and $J_3 \longrightarrow 0$.

Adding the integrals,this establishes the required results $J = 2i \sin(7\pi/2)\, I$.

**Theorem 3.3.** *The integral kernel for the wave operator* $\cos t\sqrt{-\Delta_{\nu,\mu,\eta}}$ *with triple-inverse square potential on the Euclidian plane can be written as*

$$w_{\nu,\mu,\eta}(t,p,p') = c_3 (xx')^{\nu+1/2} (yy')^{\mu+1/2} (zz')^{\eta+1/2}$$
$$\times\, t\,(t^2 - |p-p'|^2)_+^{-7/5-\nu-\mu-\eta} F_A^{(3)}(\alpha, \beta, \beta', \beta'', 2\beta, 2\beta', 2\beta'',$$
$$\frac{-4xx'}{t^2 - |p-p'|^2}, \frac{-4yy'}{t^2 - |p-p'|^2}, \frac{-4zz'}{t^2 - |p-p'|^2}), \tag{3.26}$$

*where* $F_A^{(3)}(\alpha, \beta, \beta', \beta'', 2\beta, 2\beta', 2\beta''; z, z')$ *is the three-variables Lauricella hypergeometric function* $F_A^{(3)}$ *defined in* (1.7), $\alpha = 7/5 + \nu + \mu + \eta$, $\beta = \nu + 1/2$ , $\beta' = \mu + 1/2$, $\beta'' = \eta + 1/2$ *and the constant* $c_3$ *is given by*

$$c_3 = (-1)^{\nu+\mu+\eta} 4^{2+\nu+\mu+\eta}$$
$$\times \frac{\Gamma(7/2 + \nu + \mu + \eta)\Gamma(1/2 + \nu)\Gamma(1/2 + \mu)\Gamma(1/2 + \eta)}{\pi^2 \Gamma(2\nu + 1)\Gamma(2\mu + 1)\Gamma(2\mu + 1)}. \tag{3.27}$$

*Proof.* We use essentially the formula (3.15) of Theorem 3.2, the formulas (1.2), the Fubini theorem and the formula (1.8).

## 4. Applications and Further Studies

We give an application of Theorem 3.3.

**Corollary 4.1.** *The integral kernel of the heat operator* $e^{t\Delta_{\nu,\mu,\eta}}$ *can be written in the form*

$$H_{\nu,\mu,\eta}(t,p,p') = \frac{c_4}{\sqrt{t}} (xx')^{\nu+1/2} (yy')^{\mu+1/2} (zz')^{\mu+1/2}$$
$$\times \int_{|p-p'|}^{\infty} e^{-u^2/4t} u \left(u^2 - |p-p'|^2\right)^{-7/2-\nu-\mu-\eta}$$

$$\times F_A^{(3)}\left(\alpha,\beta,\beta',\beta'',2\beta,2\beta',2\beta'',\right.$$

$$\left.-\frac{4xx'}{u^2-|p-p'|^2},-\frac{4yy'}{u^2-|p-p'|^2},-\frac{4zz'}{u^2-|p-p'|^2}\right)\,du$$

*with* $\alpha = 7/2+\nu+\mu+\eta$ ,$b=\nu+1/2$, $b'=\mu+1/2$, $\beta''=\eta+1/2$.

$$c_4 = (-1)^{\nu+\mu+\eta}4^{2+\nu+\mu+\eta}$$

$$\times\frac{\Gamma(7/2+\nu+\mu+\eta)\Gamma(1/2+\nu)\Gamma(1/2+\mu)\Gamma(1/2+\eta)}{\pi^{5/2}\Gamma(2\nu+1)\Gamma(2\mu+1)\Gamma(2\mu+1)}. \tag{4.1}$$

*Proof.* We use the transmutation formula (see Greiner et al. [8], p.362):

$$e^{t\Delta_{\nu,\mu,\eta}} = \frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-u^2/4t}\cos u\sqrt{-\Delta_{\nu,\mu,\eta}}du.$$

We suggest here a certain number of open related problems connected to this paper. For example the semi-linear wave and heat equations associated to the triple-inverse square potential and its global solution and a possible blow up of the solution in finite times.
We can also to look for the dispersive and Strichartz estimates for the Schrödinger and the wave equations with Triple-inverse square potential , for the case of inverse square potential (Burg et al. [3] and Planchon et al. [11]).

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