

ON THE DETERMINATION OF SOLUTIONS  
SETS REGARDING HIGHLY NON-LINEAR  
EQUATIONS SYSTEMS

Mahamat Saleh Daoussa Hagggar<sup>1</sup> §, Benjamin Mampassi<sup>2</sup>

<sup>1</sup>Department of Mathematics

Faculty of Exact and Applied Sciences

N'djamena University

BP: 1027, Code Postal: 00235, CHAD

<sup>2</sup>Department of Mathematics

Faculty of Science and Technology

Dakar University, BP 5005, Code Postal: 00221, SENEGAL

**Abstract:** This paper proposes an approach for determining solutions sets of larger scale non linear equations systems with more unknowns number than equations. The proposed approach consists in transforming the equations system into a one dimensional equation by using  $\alpha$ -dense curves properties. Then conditions for obtaining sets that are dense in the set of solution is established. An example is provided to demonstrate merits of our proposed algorithm.

**AMS Subject Classification:** 34A35, 35F50, 57Q55

**Key Words:** nonlinear equations systems, Newton method,  $\alpha$ -dense curves

## 1. Introduction

Suppose that we are given with the system of nonlinear equations of type

$$F(x) = 0, \tag{1}$$

where  $F = (f_1, \dots, f_p) : \mathbb{R}^n \longrightarrow \mathbb{R}^p$  is a non linear function. Such a system

---

Received: May 7, 2015

© 2015 Academic Publications

§Correspondence author

arises in many problems in sciences and engineering. This system may be formed by nonlinear algebraic, integral or transcendental equations and so on. Conditions of the existence of solution of the equation (1) are widely discussed in the literature. One can, for example, refer to [2]. Alternatively, in the case  $n = p$ , there exists a wide class of iterative methods for the numerical solution of the systems of type (1). Among them, the best-known ones is the Newton method which, starting with an initial guess, approaches a solution  $x^*$  of (1) by the following scheme

$$x^{k+1} = x^k - (F'(x^k))^{-1}F(x^k), \quad k = 0, 1, \dots, n. \quad (2)$$

Of course the use of this scheme requires at most the function  $F$  to be a continuously differentiable mapping on an open neighborhood of the solution  $x^*$ . Moreover, if the Jacobian  $F'(x^k)$  is non singular and if  $F'$  is Lipschitz continuous then the Newton method converges quadratically to  $x^*$  provided that the initial guess  $x^0$  is sufficiently closed to the solution  $x^*$ . Although the quadratic convergence is very attractive, the Newton's method suffers that its efficiency depends on a good initial approximation [1] and, also its behaving is problematic when  $F'(x^*)$  is singular. There exists a class of Newton's method versions which try to provide some amelioration for obtaining the approximation of the solution with good convergence speed. Other classes of methods for numerical solution of the equation (1) use reduction to simple one-dimensional nonlinear equations. The best-known method of this type is the nonlinear successive overrelaxation (**SOR**) method [3]. A larger variety of combined methods can be constructed, [7]. A well known example is the  $m$ -step SOR-Newton process [3, 5, 6]. All these methods do not work efficiently for the numerical solutions of (1) where  $n > p$ . In this paper a new approach for approximating the set of the solutions of (1) is investigated.

This approach is global and does not depends on initial guess as in many numerical schemes which approximate only solutions that are closed to initial guesses. The rest of this paper is organized as follows. In Section 2 we describe a numerical approach for characterizing the set of the solution of the system (1). In Section 3 we establish and prove a density theorem, and in Section 4 we illustrate this approach on a number of numerical examples.

### 2. Description of the Method

Throughout this paper,  $x$  is a  $\mathbb{R}^n$ -vector whose components are denoted by  $x_1, \dots, x_n$  and, the system (1) explicitly reads as

$$f_1(x_1, \dots, x_n) = 0, f_2(x_1, \dots, x_n) = 0, \dots, f_p(x_1, \dots, x_n) = 0. \tag{3}$$

We derive here an approach based on the idea which consists in transforming the system (3) onto a one-dimensional ones. To this end we first see that the system (3) is equivalent to the following equation

$$G(x_1, \dots, x_n) := \sum_{i=1}^p f_i^2(x_1, \dots, x_n) = 0. \tag{4}$$

Note that the equivalence of (3) and (4) is seen in the sense that all solutions of the system (3) are also solutions of the equation (4) and conversely. Next we let the set of the solution of the equation (4) to be

$$S = \{(x_1, \dots, x_n) \in \mathbb{R}^n / G(x_1, \dots, x_n) = 0\}. \tag{5}$$

Hereafter we would like to approximate the set  $S$ . Our main idea for achieving this scheme comes from the reduction transformation described in [10] that consists in considering an  $\alpha$ -dense curve of  $\mathbb{R}^n$ , that is

$$\begin{aligned} h_\alpha : \mathbb{R} &\longrightarrow \mathbb{R}^n \\ \theta &\longmapsto h_\alpha(\theta) = (h_\alpha^1(\theta), \dots, h_\alpha^n(\theta)) \end{aligned} \tag{6}$$

such that for each point  $p \in \mathbb{R}^n$ , there exists  $\theta_p \in \mathbb{R}$  and

$$\| h_\alpha(\theta_p) - p \| \leq \alpha \tag{7}$$

where  $\| \cdot \|$  denotes the  $\mathbb{R}^n$  Euclidian norm. There exists few papers from which  $\alpha$ -dense curves in  $\mathbb{R}^n$  are constructed. One can, for example, refer to [4, 8, 9, 10]. Let us set

$$G_\alpha(\theta) = G(h_\alpha(\theta)) \tag{8}$$

and define

$$S_\alpha = \{h_\alpha(\theta) \in \mathbb{R}^n / G_\alpha(\theta) = 0\}, \tag{9}$$

then it is obvious to see that

$$S_\alpha \subset S. \tag{10}$$

Now we would like to discuss about conditions that ensure the set  $S_\alpha$  to be dense in the set of solutions  $S$ . In what follows we establish this result:

**Theorem 2.1.** *Assume that*

(i)  *$S$  is the compact set,*

(ii)  *$h_\alpha$  is continuous and satisfies*

$$\lim_{\theta \rightarrow +\infty} \|h_\alpha(\theta)\| = +\infty \quad (11)$$

(iii) *and  $G$  is a  $\beta$ -Lipschitz continuous function.*

*Then for each  $P$  in  $S$ , there exists  $\hat{\theta}$  in  $\mathbb{R}$  such that*

$$\|P - h_\alpha(\hat{\theta})\| \leq \alpha \quad (12)$$

*and*

$$|G(h_\alpha(\hat{\theta}))| \leq \beta\alpha. \quad (13)$$

*Proof.* Consider  $P$  in  $S$ . As  $S$  is compact, there is a sequence  $(P_k)$  that satisfies  $G(P_k) = 0$  for each  $k$  and  $\|P - P_k\| \rightarrow 0$ . By the  $\alpha$ -dense curve property, for each  $k$  there is a real  $\theta_k$  such that

$$\|P_k - h_\alpha(\theta_k)\| \leq \alpha. \quad (14)$$

Therefore we can write

$$\|P - h_\alpha(\theta_k)\| \leq \|P - P_k\| + \alpha. \quad (15)$$

Since  $\|P - h_\alpha(\theta_k)\| \rightarrow 0$ , we deduce that the sequence  $(h_\alpha(\theta_k))$  is bounded in  $\mathbb{R}$ . By assumption (ii) it follows that the sequence  $(\theta_k)$  is bounded and consequently it admits a convergent subsequence also named by  $(\theta_k)$ . Then there is  $\hat{\theta}$  such that  $\theta_k \rightarrow \hat{\theta}$ . By continuity of  $h_\alpha$  we have

$$h_\alpha(\theta_k) \rightarrow h_\alpha(\hat{\theta}). \quad (16)$$

Also, by equations (14, 15, 16), since  $G$  is  $\beta$ -Lipschitz we have

$$|G(P_k) - G(h_\alpha(\theta_k))| \leq \beta\alpha \quad (17)$$

and passing to the limit it follows

$$|G(P) - G(h_\alpha(\hat{\theta}))| \leq \beta\alpha. \quad (18)$$

Since  $G(P) = 0$  by assumption then Theorem 1 follows.  $\square$

Observe also that Theorem 1 is sharp in the sense that, for a sufficiently little value of  $\alpha$  and where  $G$  has conveniently regularity properties then the  $S_\alpha$  is dense in  $S$  in the following sense: for each  $P$  in  $S$  there is  $\widehat{P}$  in  $S_\alpha$  such that  $\|P - \widehat{P}\| \leq \alpha$ . In such a way the conditions of the Theorem 1 permit to construct an approximation of the set of solutions of highly non linear equations systems of type (1). So that an algorithm for determining an approximation of this solution set can be summarized as follows.

- Read the discretisation parameter  $N$
- Define the function  $\theta \mapsto h_\alpha(\theta)$
- for  $k = -N$  to  $N$ 
  - determine  $\widehat{\theta}_k \in [kdt, (k + 1)dt]$  such that  $F(h_\alpha(\widehat{\theta}_k)) = 0$
  - set  $S \ni h_\alpha(\widehat{\theta}_k)$ .

### 3. An Example

There are various examples in the literature from which the set of the solutions of the system (3) by Newton methods are not very satisfactory. Here we consider in  $\mathbb{R}^3$  the following equations system

$$\begin{cases} f_1(x_1, x_2, x_3) = x_1^3 - x_1x_2x_3 = 0 \\ f_2(x_1, x_2, x_3) = x_2^2 - x_1x_3 = 0. \end{cases} \tag{19}$$

As described above we take the reduction transformation given by

$$\theta \mapsto h_\alpha(\theta) = \begin{cases} x_1 = l \cos(\theta) \\ x_2 = l \cos\left(\frac{\pi\sqrt{2}\theta}{\alpha}\right) \\ x_3 = l \cos\left(\frac{\pi^2\sqrt{6}\theta}{\alpha^2}\right). \end{cases} \tag{20}$$

It is shown in [4] that this reduction transformation is  $\alpha$ -dense in  $[-l, l]^3$ . According to Theorem 1, we define  $G : \theta \mapsto f_1^2(h_\alpha(\theta)) + f_2^2(h_\alpha(\theta))$  and we solve locally the equation  $G(\theta) = 0$  in each interval  $[kdt, (k + 1)dt]$  for  $k = -N, \dots, N$ . For our simulation we have taken  $\alpha = 0.3, l = 10$ . We used *Maltab* solver to compute the equation  $G(\theta) = 0$  in the interval  $[kdt, (k + 1)dt]$  by considering  $kdt$  as the initial guess. Then the approximated solution set in  $[-10, 10]^3$  is represented in Figure 1.

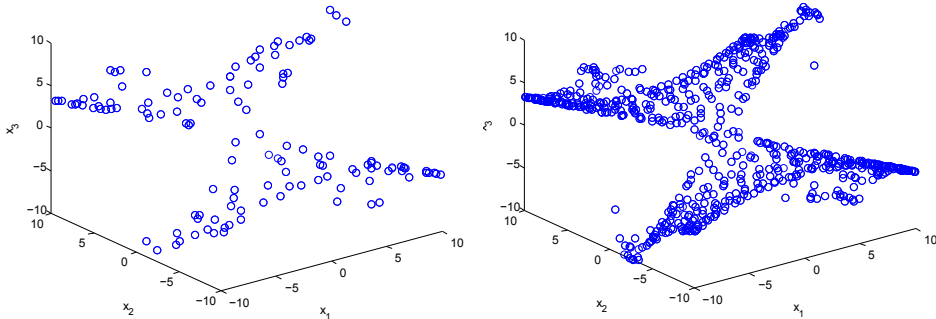


Figure 1: An illustration of the approximation of the solution set of the system (19) in  $[-10, 10]^3$  with  $dt = 0.5$  on the left and  $dt = 0.1$  on the right.

### References

- [1] D.K.R. Babajee, M.Z. Dauhoo, An analysis of the properties of the variants of Newton's method with third order convergence, *Applied Mathematics and Computation*, **183**, No 1 (2006), 659-684.
- [2] R.L. Burden, J.D. Faires, *Numerical Analysis*, 3rd Ed., Prindle, Weber and Schmidt, Boston (1985).
- [3] T.N. Grapsa, M.N. Vrahatis, A dimension-reducing method for solving systems of nonlinear equations in  $\mathbb{R}^n$ , *Inter. J. Computer Math.*, **32** (1990), 205-216.
- [4] G. Mora, Y. Cherruault, The theoretic calculation time associated with  $\alpha$ -dense curves, *Kybernetes*, **27**, No 8-9 (1998), 919-939.
- [5] J.M. Ortega, W.C. Rheinbolt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York (1970).
- [6] W.C. Rheinboldt, *Methods for Solving Systems of Equations*, SIAM, Philadelphia (1974).
- [7] A.R. Soheili, S.A. Ahmadian, J. Naghipoor, A family of predictor-corrector methods based on weight combinations of quadratures for solving nonlinear equations, *International Journal of Nonlinear Science*, **6** (2008), 29-33.
- [8] A. Ziadi, Y. Cherruault, G. Mora, Global optimization, a new variant of the Alienor method, *Comp. Math. Appl.*, **41** (2001), 63-71.

- [9] A. Ziadi, Y. Cherruault, Generation of  $\alpha$ -dense curves and applications to global optimization, *Kybernetes*, **29**, No 1 (2000), 71-82.
- [10] A. Ziadi, Y. Cherruault, G. Mora, The existence of  $\alpha$ -dense curves with minimal length in a metric space, *Kybernetes*, **29**, No 2 (2000), 219-230.

