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3-POINT NONLINEAR TERNARY INTERPOLATING SUBDIVISION SCHEMES

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Abstract: It is a well known fact that smooth curves generated by linear interpolating schemes produce Gibbs phenomenon or oscillations near irregular initial data points. Main aim of this article to introduce some nonlinear subdivision scheme which is convergent, keeps all initial data points and eliminates Gibbs phenomenon. We have introduced a new class of 3-point nonlinear ternary interpolating subdivision schemes which has these properties. Numerical results are presented to support our claim.

AMS Subject Classification: 41A05, 41A10, 65D17, 65D05
Key Words: interpolating subdivision scheme, Gibbs phenomenon, convergence, smoothness, nonlinear subdivision scheme, oscillatory limit curve

1. Introduction

Subdivision schemes have been widely used to generate smooth curves and surfaces from initial data points. Interpolating and approximating subdivision schemes are two major categories of subdivision schemes. In interpolating subdivision schemes, more data points are added between the initial or existing data points at each level of subdivision. However, in approximating subdivision schemes existing points are replaced by their approximations and new points are inserted between them at each level of refinement. As a result, curves generated by approximating schemes are smoother but do not pass through the given initial data points especially at and near larger jumps or discontinuities.

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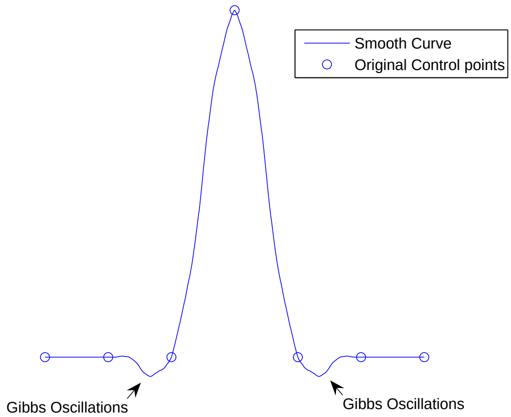


Figure 1: Initial control points and curve generated by 3-point ternary interpolating subdivison scheme (5th level) in (3.1) with $w = \frac{5}{18}$.

On the other hand curves generated by interpolating subdivision schemes passes through the given initial data points but produces oscillations also known as Gibbs Phenomenon near the points with large jumps or discontinuities. Graphic explanation of Gibbs Phenomenon is given in Figure (1).

Gibbs phenomenon in curves, generated by interpolating subdivision schemes is undesirable for some applications. Nonlinear subdivision Schemes like ENO, WENO, PPH and regularization strategies ([1], [2], [3], [4], [5], [6], [9], [10]) were introduced during last several years to address this oscillation phenomenon. More recently, Amat et al. ([1], [2]) introduced PPH 4-points binary and ternary nonlinear interpolating subdivision schemes. Arithmetic mean of second differences was replaced by their harmonic mean in linear subdivision scheme to change it to nonlinear scheme. Our ternary nonlinear interpolating schemes are inspired by the approach presented in ([1], [2]) but different in three ways. First, we used geometric mean instead of harmonic mean, second, our schemes are odd points ternary instead of even points and third, we used first differences instead of second differences.

For better understanding of this article, we have arranged the material in following fashion. In Section 2, preliminary concepts and their properties along with some basic terminology are discussed, in Section 3, nonlinear interpolating subdivision schemes are introduced. The convergence of these schemes is analyzed in Section 4 and some numerical results are presented in Section 5.

2. Preliminaries

A general form of $(2n-1)$-points linear univariate ternary subdivision scheme S which maps set of data points $f^k = \{f_i^k\}_{i \in Z}$ into the next refinement level of data points $f^{k+1} = \{f_i^{k+1}\}_{i \in Z}$ is defined as

$$\begin{aligned} f_{3i-1}^{k+1} &= \sum_{j=-(n-1)}^{n-1} a_j f_{i+j}^k, \\ f_{3i}^{k+1} &= \sum_{j=-(n-1)}^{n-1} b_j f_{i+j}^k, \\ f_{3i+1}^{k+1} &= \sum_{j=-(n-1)}^{n-1} a_{-j} f_{i+j}^k. \end{aligned} \tag{2.1}$$

Above equation can also be expressed as $f^{k+1} = Sf^k$. A necessary condition for the uniform convergence of ternary subdivision scheme (2.1) given by [8] is

$$\sum_{j=-(n-1)}^{n-1} a_j = \sum_{j=-(n-1)}^{n-1} b_j = \sum_{j=-(n-1)}^{n-1} a_{-j} = 1. \tag{2.2}$$

By defining b_j as

$$b_j = \begin{cases} 1 & \text{for} \quad j = 0, \\ 0 & \text{for} \quad j \neq 0, \end{cases} \tag{2.3}$$

then equation (2.1) becomes

$$\begin{aligned} f_{3i-1}^{k+1} &= \sum_{j=-(n-1)}^{n-1} a_j f_{i+j}^k, \\ f_{3i}^{k+1} &= f_i^k, \\ f_{3i+1}^{k+1} &= \sum_{j=-(n-1)}^{n-1} a_{-j} f_{i+j}^k. \end{aligned} \tag{2.4}$$

Subdivision scheme given in equation (2.4) is called interpolating subdivision scheme.

For $(x, y) \in R^2$, we define a nonlinear function called Modified Geometric Mean or MGM as

$$MGM(x, y) = \begin{cases} sign(x)\sqrt{xy} & \text{if} \quad xy > 0, \\ 0 & \text{if} \quad xy \leq 0, \end{cases} \tag{2.5}$$

where $sign(x) = 1$ if $x \geq 0$ and $sign(x) = 0$ if $x < 0$. Nonlinear function MGM defined above has several interesting properties like,

$$MGM(x,y) = MGM(y,x), \tag{2.6}$$

$$MGM(-x,-y) = -MGM(x,y), \tag{2.7}$$

$$|MGM(x,y)| \leq \max(|x|,|y|). \tag{2.8}$$

We recall PPH function $(x,y) \in R^2$ defined by ([1], [2]) as

$$PPH(x,y) = \begin{cases} (1 + sign(xy)\frac{xy}{x+y} & \text{for} \quad xy > 0, \\ 0 & \text{if} \quad xy \leq 0. \end{cases} \tag{2.9}$$

PPH function defined above also satisfy the properties in (2.6), (2.7) and (2.8).

3. 3-point Nonlinear Ternary Subdivision Schemes

We start with a well known linear 3-point ternary interpolating subdivision scheme,

$$\begin{aligned} f_{3i-1}^{k+1} &= wf_{i-1}^k + (\tfrac{4}{3} - 2w)f_i^k + (w - \tfrac{1}{3})f_{i+1}^k, \\ f_{3i}^{k+1} &= f_i^k, \\ f_{3i+1}^{k+1} &= (w - \tfrac{1}{3})f_{i-1}^k + (\tfrac{4}{3} - 2w)f_i^k + wf_{i+1}^k. \end{aligned} \tag{3.1}$$

Which is C^1 for $\frac{2}{9} < w < \frac{1}{3}$ as proved by Hassan etl [7]. We define $df_i = f_{i+1} - f_i$, and rewrite above scheme as

$$\begin{aligned} f_{3i-1}^{k+1} &= (2w - \tfrac{1}{3})f_{i-1}^k + (\tfrac{4}{3} - 2w)f_i^k + 2(w - \tfrac{1}{3})(\tfrac{df_i^k + df_{i-1}^k}{2}), \\ f_{3i}^{k+1} &= f_i^k, \\ f_{3i+1}^{k+1} &= (\tfrac{4}{3} - 2w)f_i^k + (2w - \tfrac{1}{3})f_{i+1}^k - 2(w - \tfrac{1}{3})(\tfrac{df_i^k + df_{i-1}^k}{2}). \end{aligned} \tag{3.2}$$

Replacing the arithmetic mean $\frac{df_i^k + df_{i-1}^k}{2}$ in the above equation (3.2) by modified geometric mean $MGM(df_i^k, df_{i-1}^k)$ as defined in (2.5), we get a class of nonlinear 3-point ternary interpolating schemes,

$$\begin{aligned} f_{3i-1}^{k+1} &= (2w - \tfrac{1}{3})f_{i-1}^k + (\tfrac{4}{3} - 2w)f_i^k + 2(w - \tfrac{1}{3})MGM(df_i^k, df_{i-1}^k), \\ f_{3i}^{k+1} &= f_i^k, \\ f_{3i+1}^{k+1} &= (\tfrac{4}{3} - 2w)f_i^k + (2w - \tfrac{1}{3})f_{i+1}^k - 2(w - \tfrac{1}{3})MGM(df_i^k, df_{i-1}^k). \end{aligned} \tag{3.3}$$

Similarly, if we replace arithmetic mean $\frac{df_i^k + df_{i-1}^k}{2}$ in equation (3.2) by modified harmonic mean also known as PPH function, $PPH(df_i^k, df_{i-1}^k)$ as defined in (2.9), we get another class of nonlinear 3-point ternary interpolating schemes.

$$\begin{aligned} f_{3i-1}^{k+1} &= (2w-\tfrac{1}{3})f_{i-1}^k + (\tfrac{4}{3}-2w)f_i^k + 2(w-\tfrac{1}{3})PPH(df_i^k, df_{i-1}^k), \\ f_{3i}^{k+1} &= f_i^k, \\ f_{3i+1}^{k+1} &= (\tfrac{4}{3}-2w)f_i^k + (2w-\tfrac{1}{3})f_{i+1}^k - 2(w-\tfrac{1}{3})PPH(df_i^k, df_{i-1}^k). \end{aligned} \tag{3.4}$$

4. Convergence of Nonlinear subdivision schemes

Nonlinear schemes (3.3) can be expressed as

$$f^{k+1} = S_{NL}(f^k) = S(f^k) + F(df^k), \tag{4.1}$$

where S_{NL} is representing nonlinear subdivison scheme, S is linear interpolating subdivision scheme given by

$$\begin{aligned} S(f^k)_{3i-1} &= (2w-\tfrac{1}{3})f_{i-1}^k + (\tfrac{4}{3}-2w)f_i^k, \\ S(f^k)_{3i} &= f_i^k, \\ S(f^k)_{3i+1} &= (\tfrac{4}{3}-2w)f_i^k + (2w-\tfrac{1}{3})f_{i+1}^k \end{aligned} \tag{4.2}$$

and $F(df)$ is given by

$$\begin{aligned} F(df^k)_{3i-1} &= 2(w-\tfrac{1}{3})MGM(df_i^k, df_{i-1}^k), \\ F(df^k)_{3i} &= 0, \\ F(df^k)_{3i+1} &= -2(w-\tfrac{1}{3})MGM(df_i^k, df_{i-1}^k). \end{aligned} \tag{4.3}$$

It can be easily seen by simple criterion given in [7] that subdivision scheme S above in (4.2) is convergent and has C^0 continuity for $\frac{1}{6} < w < \frac{2}{3}$.

To prove the convergence of nonlinear scheme S_{NL}, we recall following result from ([1], [2]).

Theorem 4.1. *For F, S and d given in (4.1) if $\exists M > 0$ such that $\forall g \in l^\infty(Z)$*

$$||F(g)||_\infty \leq M||g||_\infty, \tag{4.4}$$

and $\exists c < 1$ such that

$$||dS(f) + dF(df)||_\infty \leq c||df||_\infty, \tag{4.5}$$

then the subdivision scheme S_{NL} is uniformly convergent. Moreover, if S is C^α convergent then, for all sequence $f \in l^\infty(Z)$, $S_{NL}^\infty(f)$ is at least C^β with $\beta = \min(\alpha, -\log_2(c))$.

Since ternary subdivision has three legs, so in order to prove (4.4) and (4.5) for our nonlinear scheme, we have to consider each of them one by one. First at $n = 3i - 1$,

$$|dS(f)_{3i-1} + dF(df)_{3i-1}| = |S(f)_{3i} - S(f)_{3i-1} + F(df)_{3i} - F(df)_{3i-1}|$$

$$= |f_i - (2w - \frac{1}{3})f_{i-1} - (\frac{4}{3} - 2w)f_i - 2(w - \frac{1}{3})MGM(df_i, df_{i-1})|$$

$$= | - (2w - \frac{1}{3})f_{i-1} + (2w - \frac{1}{3})f_i - 2(w - \frac{1}{3})MGM(df_i, df_{i-1})|$$

$$= |(2w - \frac{1}{3})(f_i - f_{i-1}) - 2(w - \frac{1}{3})MGM(df_i, df_{i-1})|$$

$$= |(2w - \frac{1}{3})df_{i-1} + (\frac{2}{3} - 2w)MGM(df_i, df_{i-1})|$$

$$\leq (2w - \frac{1}{3})|df_{i-1}| + (\frac{2}{3} - 2w)|MGM(df_i, df_{i-1})|.$$

Here we restricting $\frac{1}{6} < w < \frac{1}{3}$ to get $(2w - \frac{1}{3}) \geq 0$ and $(\frac{2}{3} - 2w) \geq 0$. Since $|df_{i-1}| \leq \max(|df_i|)$ and $|MGM(df_i, df_{i-1})| \leq \max(|df_i|)$, therefore, we have

$$|dS(f)_{3i-1} + dF(df)_{3i-1}| \leq (2w - \frac{1}{3} + \frac{2}{3} - 2w)\max(|df_i|)$$

$$|dS(f)_{3i-1} + dF(df)_{3i-1}| \leq \frac{1}{3}\max(|df_i|). \tag{4.6}$$

Next we consider $n = 3i$,

$$|dS(f)_{3i} + dF(df)_{3i}| = |S(f)_{3i+1} - S(f)_{3i} + F(df)_{3i+1} - F(df)_{3i}|$$

$$= |(\frac{4}{3} - 2w)f_i + (2w - \frac{1}{3})f_{i+1} - f_i - 2(w - \frac{1}{3})MGM(df_i, df_{i-1})|$$

$$= |(\frac{1}{3} - 2w)f_i + (2w - \frac{1}{3})f_{i+1} - 2(w - \frac{1}{3})MGM(df_i, df_{i-1})|$$

$$= |(2w - \frac{1}{3})df_i - 2(w - \frac{1}{3})MGM(df_i, df_{i-1})|.$$

Again $\frac{1}{6} < w < \frac{1}{3}$ to get $(2w - \frac{1}{3}) \geq 0$ and $(\frac{2}{3} - 2w) \geq 0$ and using the facts $|df_i| \leq \max(|df_i|)$ and $|MGM(df_i, df_{i-1})| \leq \max(|df_i|)$, we get

$$|dS(f)_{3i} + dF(df)_{3i}| \leq \frac{1}{3}\max(|df_i|). \tag{4.7}$$

At $n = 3i + 1$,

$$|dS(f)_{3i+1} + dF(df)_{3i+1}| = |S(f)_{3i+2} - S(f)_{3i+1} + F(df)_{3i+2} - F(df)_{3i+1}|$$
$$=|(2w - \frac{1}{3})f_i + (\frac{4}{3} - 2w)f_{i+1} + 2(w - \frac{1}{3})MGM(df_{i+1}, df_i)$$
$$- (\frac{4}{3} - 2w)f_i - (2w - \frac{1}{3})f_{i+1} + 2(w - \frac{1}{3})MGM(df_i, df_{i-1})|$$
$$=|(4w - \frac{5}{3})f_i + (\frac{5}{3} - 4w)f_{i+1} + 2(w - \frac{1}{3})(MGM(df_{i+1}, df_i)$$
$$+ MGM(df_i, df_{i-1}))|$$
$$=|(\frac{5}{3} - 4w)df_i + 2(w - \frac{1}{3})(MGM(df_{i+1}, df_i) + MGM(df_i, df_{i-1}))|.$$

For $w < \frac{1}{3}$,

$$|dS(f)_{3i+1} + dF(df)_{3i+1}|$$
$$\leq (\frac{5}{3} - 4w)|df_i| + 2(\frac{1}{3} - w)(|MGM(df_{i+1}, df_i)| + |MGM(df_i, df_{i-1}))|.$$

Since $|MGM(df_{i+1}, df_i)| \leq \max |df_i|$, so,

$$|dS(f)_{3i+1} + dF(df)_{3i+1}| \leq \{(\frac{5}{3} - 4w) + 4(\frac{1}{3} - w)\}(\max(|df_i|)).$$

This implies

$$|dS(f)_{3i+1} + dF(df)_{3i+1}| \leq c \ \max(|df_i|), \tag{4.8}$$

where $c = 3 - 8w < 1$ for $\frac{1}{4} < w < \frac{1}{3}$.

Equations (4.6), (4.7) and (4.8) together give,

$$\max_n |dS(f)_n + dF(df)_n| \leq c \ \max_n(|df_n|),$$

or

$$||dS(f) + dF(df)||_\infty \leq c||df||_\infty, \tag{4.9}$$

for $c < 1$ with $\frac{1}{4} < w < \frac{1}{3}$. This proves equation (4.5).

Now to prove equation (4.4), we consider $F(g)$ at $n = 3i - 1$,

$$|F(g)_{3i-1}| \leq 2(\frac{1}{3} - w)\max(|g_i|) \tag{4.10}$$

for $w < \frac{1}{3}$ and by using property (2.8).

At $n = 3i$,

$$|F(g)_{3i}| = 0 \leq M \max(|g_i|) \tag{4.11}$$

for any $M > 0$ and similarly, at $n = 3i + 1$,

$$|F(g)_{3i+1}| \leq 2(\frac{1}{3} - w) \max(|g_i|) \tag{4.12}$$

for $w < \frac{1}{3}$ and by using property (2.8).

Equations (4.10), (4.11) and (4.12) give

$$\max_n |F(g)_n| \leq M \max_n(|g_n|),$$

or

$$||F(g)||_\infty \leq M||g||_\infty, \tag{4.13}$$

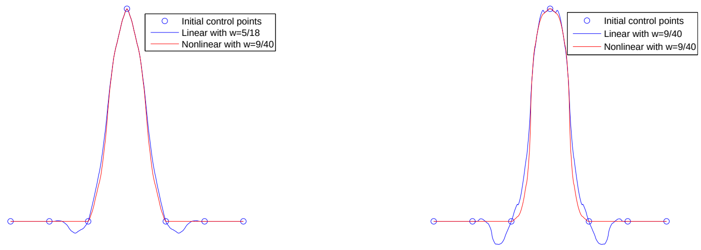
with $M = \frac{1}{6}$ for $\frac{1}{4} < w < \frac{1}{3}$. Which proves equation (4.4) and consequently proves that our class of nonlinear 3-point ternary interpolating subdivision schemes S_{NL} given in(3.3) is uniformly convergent for $\frac{1}{4} < w < \frac{1}{3}$. Theorem 4.1 also proves that S^∞_{NL} is at least C^0.

Since the functions PPH and MGM are nonlinear and satisfy properties given in (2.6), (2.7) and (2.8) which we used for MGM to prove the convergence of (3.3). Therefore, it can easily be verified by replacing MGM function with PPH function in the above proof that subdivision schemes (3.4) is also convergent.

5. Numerical Results

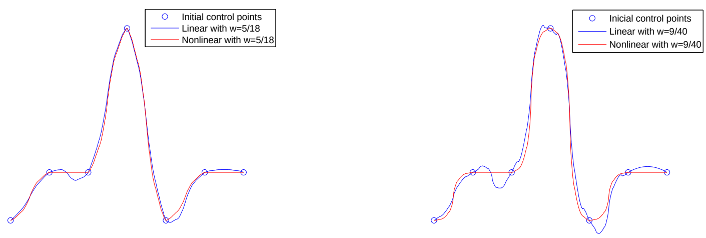
We picked two examples with varying number of irregularities in initial data points. In Figure (2)(a), two smooth curves are generated from the initial control or data points, one with linear scheme (3.1) and another with a nonlinear scheme (3.3) both with $k = \frac{5}{18}$. One can easily see oscillations or Gibbs phenomenon near the initial points with bigger jumps for linear scheme but for nonlinear scheme it is eliminated. Figure (2)(b) also shows two curves one with the linear scheme (3.1) and one with nonlinear scheme (3.3) with $k = \frac{9}{40}$ on the same initial data points. The improvement is evident. The curves generated from nonlinear scheme are free of Gibbs phenomenon.

The second example with more irregular data points is shown in Figure(3) (a) and (b). The curves are generated by 3-point linear interpolating subdivision scheme (3.1) with $w = \frac{5}{18}$ and by nonlinear interpolating subdivision scheme (3.3) with $w = \frac{5}{18}$ in (3)(a) and $w = \frac{9}{40}$ in (3)(b) respectively. For all these curves, we used 5-levels of subdivisions. Both these examples show impressive



(a) (b)

Figure 2: (a) Initial control points and the curves generated by 3-point linear and nonlinear ternary interpolating subdivision schemes at 5th level with $k = \frac{5}{18}$ in (3.1)and (3.3) respectively. (b) Initial control points and the curves generated by 3-point linear and nonlinear ternary interpolating subdivision schemes at 5th level with $k = \frac{9}{40}$ in (3.1)and (3.3) respectively.



(a) (b)

Figure 3: (a) Comparison at 5th level of linear scheme with $w = \frac{5}{18}$ in (3.1) is given with nonlinear scheme with $w = \frac{5}{18}$ in (3.3). (b) Again comparison at 5th level of linear scheme with $w = \frac{9}{40}$ in (3.1) is given with nonlinear scheme with $w = \frac{9}{40}$ in (3.3).

improvement in eliminating oscillations or Gibbs phenomenon.

6. Conclusion

In this article, we introduced a class of 3-point nonlinear interpolating subdivision schemes. It is proved that our schemes are convergent. Numerical results are presented to show that curves generated by these schemes are free of Gibbs phenomenon.

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