International Journal of Applied Mathematics

Volume 28 No. 4 2015, 415-425
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v28i4.9

SOME RESULTS ON THE CLASS OF α-CONVEX JANOWSKI TYPE FUNCTIONS AND CLASS $\mathcal{U}$

Edmond Aliaga[1], Nikola Tuneski[2 §]

[1]Department of Mathematics and Computer Sciences
Faculty of Mathematical and Natural Sciences
University of Prishtina
M. Tereza n.n. 10000 Prishtina, KOSOVO
[2]Faculty of Mechanical Engineering
Ss. Cyril and Methodius University in Skopje
Karpoš II b.b., 1000 Skopje, REPUBLIC OF MACEDONIA

Abstract: Using methods from the theory of differential subordinations, we obtain several results that describe the relation between two classes univalent functions. Examples that demonstrate these results are provided.

AMS Subject Classification: 30C45
Key Words: analytic function, starlike function, sufficient condition, differential subordination

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ that are normalized such that $f(0) = f'(0) - 1 = 0$.

The class of *starlike functions of order* α, $0 \le \alpha < 1$, is defined by

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha, z \in \mathbb{D} \right\}.$$

For $\alpha = 0$ we receive the class of *starlike functions* $S^* \equiv S^*(0)$ consisting of

Received: May 24, 2015

§Correspondence author

functions f that map the unit disk onto a starlike region, i.e., if $\omega \in f(\mathbb{D})$, then $t\omega \in f(\mathbb{D})$ for all $t \in [0,1]$.

Next, we denote by

$$\mathcal{K}^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha, z \in \mathbb{D} \right\}$$

the class of *convex functions of order* α, $0 \leq \alpha < 1$. Here, $K^* \equiv K^*(0)$ is the class of *convex functions* such that $f \in \mathcal{K}$ if and only if $f(\mathbb{D})$ is a convex region, i.e., if for any $\omega_1, \omega_2 \in f(\mathbb{D})$ follows $t\omega_1 + (1-t)\omega_2 \in f(\mathbb{D})$ for all $t \in [0,1]$. All the above classes are in the class of univalent functions in $\mathbb{D}$ and even more, $\mathcal{K} \subset \mathcal{S}^*$. For details, see [2].

Now, let A and B be real numbers such that $-1 \leq B < A \leq 1$. Then, the function $(1+Az)/(1+Bz)$ maps the unit disk univalently onto an open disk that lies in the right half of the complex plane and is centered on the real axis with diameter end points $(1-A)/(1-B)$ and $(1+A)/(1+B)$. Thus, classes $\mathcal{S}^*$ and $\mathcal{K}$ can be generalized with

$$\mathcal{S}^*[A,B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right\}$$

and

$$\mathcal{K}[A,B] = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \right\}.$$

Functions in $\mathcal{S}^*[A,B]$ and $\mathcal{K}[A,B]$ are called Janowski starlike and Janowski convex functions, respectively ([4]).

Using the operators

$$J(f,\alpha;z) \equiv (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \quad (\alpha \in \mathbb{R})$$

and

$$U(f,\mu;z) = \left(\frac{z}{f(z)}\right)^{1+\mu} \cdot f'(z) \quad (\mu \in \mathbb{C}),$$

we can define classes

$$\mathcal{M}[A,B,\alpha] = \left\{ f \in \mathcal{A} : J(f,\alpha;z) \prec \frac{1+Az}{1+Bz} \right\} \quad (-1 \leq B \leq A \leq 1)$$

and

$$\mathcal{U}(\lambda,\mu) = \left\{ f \in \mathcal{A} : \frac{z}{f(z)} \neq 0 \text{ and } |U(f,\mu;z) - 1| < \lambda,\ z \in \mathbb{D} \right\} \quad (\mu \in \mathbb{C}, \lambda > 0).$$

The class $\mathcal{M}[A,B,\alpha]$ is called class of *α-convex Janowski type functions*. Well known results are that $\mathcal{M}[A,B,\alpha] \subset \mathcal{M}[A,B,\beta] \subset \mathcal{M}[A,B,0] = \mathcal{S}^*[A,B]$ for $0 \le \alpha/\beta \le 1$, and $\mathcal{M}[A,B,\alpha] \subset \mathcal{M}[A,B,1] = \mathcal{K}[A,B]$. So, class $\mathcal{M}[A,B,\alpha]$ makes a "bridge" between the classes of Janowski starlike and Janowski convex functions and consists of univalent functions.

The other class, class $\mathcal{U}(\lambda,\mu)$ and its special cases $\mathcal{U}(\lambda) \equiv \mathcal{U}(\lambda,1)$ and $\mathcal{U} \equiv \mathcal{U}(1) = \mathcal{U}(1,1)$ are also widely studied in the past decades ([1], [3], [8]-[16]). It is known [1, 16] that functions in $\mathcal{U}(\lambda)$ are univalent if $0 < \lambda \le 1$, but not necessarily univalent if $\lambda > 1$. Its importance raises from the fact that $\mathcal{U}(\lambda,\mu)$ is not a subset of $\mathcal{S}^*$ which is rear case in the theory of univalent functions. An example for such behaviour is the function

$$f(z) = \frac{z}{1+\frac{1}{2}z+\frac{1}{2}z^3} \in \mathcal{U} \setminus \mathcal{S}^*.$$

The aim of this paper is to study the relation between the classes $\mathcal{M}[A,B,\alpha]$ and $\mathcal{U}(\lambda,\mu)$. For that purpose we will use methods from the theory of first order differential subordinations.

Let $f(z)$ and $g(z)$ be analytic in the unit disk $\mathbb{D}$. Then we say that $f(z)$ is subordinate to $g(z)$, and we write $f(z) \prec g(z)$, if $g(z)$ is univalent in $\mathbb{D}$, $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. Further, we use the method of differential subordination introduced by Miler and Mocanu ([5],[6]). In fact, if $\phi : \mathbb{C}^2 \to \mathbb{C}$ ($\mathbb{C}$ is the complex plane) is analytic in domain $D \subset \mathbb{C}$, if $h(z)$ is univalent in $\mathbb{D}$, and $p(z)$ is analytic in $\mathbb{D}$ with $(p(z), zp'(z)) \in D$, when $z \in \mathbb{D}$, then we say that $p(z)$ satisfies a first-order differential subordination if

$$\phi(p(z), zp'(z)) \prec h(z). \tag{1}$$

The univalent function $q(z)$ is called a dominant of the differential subordination (1) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1). If $\tilde{q}(z)$ is a dominant of (1) and $\widetilde{q}(z) \prec q(z)$ for all dominants of (1), then we say that $\tilde{q}(z)$ is the best dominant of the differential subordination (1).

We will make use of the following lemma.

Lemma 1. *[7] Let q be univalent in the unit disk $\mathbb{D}$, and let $\theta(w)$ and $\phi(w)$ be analytic in a domain D containing $q(\mathbb{D})$, with $\phi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that:*

(i) Q is starlike in the unit disk $\mathbb{D}$,

(ii) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} = \operatorname{Re}\left[\dfrac{\theta'(q(z))}{\phi(q(z))} + \dfrac{zQ'(z)}{Q(z)}\right] > 0,\ \ z \in \mathbb{D}.$

If p is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z) \tag{2}$$

then $p(z) \prec q(z)$, and q is the best dominant of (2).

2. Main Results and Consequences

The following theorem will lead us to certain connections between the class $\mathcal{M}[A, B, \alpha]$ and the class $\mathcal{U}(\lambda, \mu)$.

Theorem 1. *Let $f \in \mathcal{A}$, $-1 \leq B < A < 1$ and $\alpha \neq 0$. Also, let $f'(z) \neq 0$ for all $z \in \mathbb{D}$. If*

$$J(f, \alpha; z) \prec \frac{1 + Az}{1 + Bz} \equiv h(z), \tag{3}$$

then

$$U(f, -1/\alpha; z) \prec q(z) \equiv \begin{cases} (1 + Bz)^{c_1}, & \text{if } B \neq 0 \text{ and } \arcsin|B| \leq \pi/|c_1| \\ e^{Az/\alpha}, & \text{if } B = 0 \end{cases}, \tag{4}$$

where $c_1 = \frac{A-B}{\alpha B}$ and $c_2 = \frac{A}{\alpha}$. The function $q(z)$ is the best dominant of (3).

Proof. Let $p(z) = U(f, -1/\alpha; z) = \left(\frac{z}{f(z)}\right)^{1-1/\alpha} \cdot f'(z)$, $q(z) = (1 + Bz)^c$, $\theta(\omega) = 1$ and $\phi(\omega) = \frac{\alpha}{\omega}$. Now we will prove that all the conditions of Lemma 1 hold.

First, let note that $q(z)$ is analytic in $\mathbb{D}$ and next we will show that it is also univalent. If $B = 0$, i.e. for $q(z) = e^{Az/\alpha}$ it is obvious. Further, let consider the case $B \neq 0$. In the beginning note that the condition $\arcsin|B| \leq \pi/|c_1|$ implies

$$|\arg q(z)| = |c_1| \cdot |\arg(1 + Bz)| < |c_1| \cdot \arcsin|B| \leq \pi, \quad z \in \mathbb{D}$$

which ensures that for any $z \in \mathbb{D}$, z and $q(z)$ lie on the same side of the real axis. So, if z_1 and z_2 lie on different sides of the real axis ($z_1 \neq z_2$), then so do $q(z_1)$ and $q(z_2)$, i.e. $q(z_1) \neq q(z_2)$. If $z_1 \neq z_2$ and they are on the same side of the real axis, then

$$z_1 \neq z_2 \quad \Rightarrow \quad 1 + Bz_1 \neq 1 + Bz_2 \quad \Rightarrow \quad q(z_1) \neq q(z_2).$$

This completes the proof that $q(z)$ is an univalent function.

Further, functions $\theta(\omega)$ and $\phi(\omega)$ are analytic with domain $D \equiv \mathbb{C} \setminus \{0\}$ containing $q(\mathbb{D})$, and $q(\mathbb{D})$ does not contain the origin. Also, $\phi(\omega) \neq 0$ when $\omega \in q(\mathbb{D})$, since $\alpha \neq 0$.

The conditions (i) and (ii) from Lemma 1 hold since

$$Q(z) = zq'(z)\,\phi(q(z)) = \alpha \frac{zq'(z)}{q(z)} = \frac{(A-B)\,z}{1+Bz},$$

$h(z) = 1 + Q(z)$ and $zh'(z)/Q(z) = zQ'(z)/Q(z) = \frac{1}{1+Bz}$.

Next, $p(z)$ is analytic in $\mathbb{D}$ and $p(0) = q(0) = 1$. Also, $p(z) \neq 0$ for all $z \in \mathbb{D}$, i.e. $p(\mathbb{D}) \subseteq D$, since $f'(z) \neq 0$ for all $z \in \mathbb{D}$ (condition of the theorem); $z/f(z) = 1 \neq 0$ for $z = 0$ (because $f \in \mathcal{A}$) and $f(z)$ has no poles on $\mathbb{D}$.

Therefore, from Lemma 1 and the fact that

$$\begin{aligned} 1 + \alpha \cdot \frac{zp'(z)}{p(z)} &= \theta(p(z)) + zp'(z)\phi(p(z)) = J(f, \alpha; z) \\ &\prec \frac{1+Az}{1+Bz} = \theta(q(z)) + zq'(z)\phi(q(z)), \end{aligned}$$

we receive $p(z) \prec q(z)$, i.e. the relation (4), and we also receive that $q(z)$ is the best dominant of (3). □

Using the definition of subordination we obtain a corollary with additional information about the relation between the classes $\mathcal{U}(\lambda, \mu)$ and $\mathcal{M}[A, B, \alpha]$.

Corollary 1. *Let $f \in \mathcal{A}$, $-1 \leq B < A \leq 1$ and $\alpha \in \mathbb{R} \setminus \{0\}$. If $f \in \mathcal{M}[A, B, \alpha]$ and $f'(z) \neq 0$ for all $z \in \mathbb{D}$, then*

$$U(f, -1/\alpha; z) \prec q(z) \equiv \begin{cases} (1+Bz)^{c_1}, & \text{if } B \neq 0 \text{ and } \arcsin|B| \leq \pi/|c_1| \\ e^{Az/\alpha}, & \text{if } B = 0 \end{cases},$$

where $c_1 = \frac{A-B}{\alpha B}$ and $c_2 = \frac{A}{\alpha}$. Additionally, if $z/f(z) \neq 0$ for all $z \in \mathbb{D}$, $A - B \leq |\alpha|$, then

$$f \in \mathcal{U}(\lambda, -1/\alpha),$$

where

$$\lambda = \begin{cases} (1+|B|)^{c_1} - 1, & B\alpha > 0 \\ (1-|B|)^{c_1} - 1, & B\alpha < 0 \\ e^{|c_2|} - 1, & B = 0 \end{cases}.$$

This implication is sharp, i.e., the constant λ can not be replaced by a smaller one such that the implication still holds.

Proof. In the beginning, let note that from the definition of the class $\mathcal{M}[A, B, \alpha]$

$$J(f, \alpha; z) \prec \frac{1 + Az}{1 + Bz} \tag{5}$$

and that all other conditions of Theorem 1 hold. So,

$$U(f, -1/\alpha; z) \prec q(z)$$

and $q(z)$ is the best dominant of (5). Further, in the proof of Theorem 1 it was shown that $q(z)$ is an univalent function. Therefore, by the definition of subordination, we have that

$$U(f, -1/\alpha; z) \in q(\mathbb{D}), \quad z \in \mathbb{D},$$

and in order to prove the second implication of this corollary, it is enough to show that

$$\lambda = \sup\{|q(z) - 1| : z \in \mathbb{D}\}.$$

Indeed, the function

$$q_1(z) = \begin{cases} \frac{q(z)-1}{Bc_1}, & B \neq 0 \\ \frac{q(z)-1}{c_1}, & B = 0 \end{cases}$$

is symmetric with respect to the real axis $\left(q_1(\overline{z}) = \overline{q_1(z)}\right)$ and it is also in the class of convex functions $\mathcal{K}$ since

$$1 + \frac{zq_1''(z)}{q_1'(z)} = \begin{cases} \frac{1+Bc_1z}{1+Bz}, & B \neq 0 \\ 1 + c_2z, & B = 0 \end{cases}$$

and

$$A - B \leq \alpha \quad \Rightarrow \quad \begin{cases} |Bc_1| \leq 1, & B \neq 0 \\ |c_2| \leq 1, & B = 0 \end{cases}.$$

Therefore,

$$\sup\{|q(z) - 1| : z \in \mathbb{D}\} = \max\{q(1) - 1, q(-1) - 1\}$$

which is easy to be verified that is equal to λ.

The sharpness of the second implication of the corollary follows from the fact that $q(z)$ is the best dominant of (5) and the definition of the best dominant. Namely, if $f \in \mathcal{U}(\lambda_1, -1/\alpha)$ for some λ_1, then

$$U(f, -1/\alpha; z) \prec 1 + \lambda_1 z$$

which implies $q(z) \prec 1 + \lambda_1 z$, i.e., $\lambda \leq \lambda_1$. □

Remark 1. In Corollary 1, only for simplicity of the results, α is taken to be real (non zero) number. The corollary can be extended to $\alpha \in \mathbb{C} \setminus \{0\}$.

Choosing $A = (1+\alpha)\lambda$ and $B = \lambda$ in Theorem 1 we receive the following result.

Corollary 2. *Let $f \in \mathcal{A}$, $0 < \lambda \leq 1$ and $0 < \alpha \leq \frac{1}{\lambda} - 1$. Also, let $f'(z) \neq 0$ for all $z \in \mathbb{D}$. If $f \in \mathcal{M}[(1+\alpha)\lambda, \lambda, \alpha]$, or equivalently,*

$$J(f, \alpha; z) \prec 1 + \alpha - \frac{\alpha}{1 + \lambda z},$$

then

$$U(f, -1/\alpha; z) \prec 1 + \lambda z,$$

or equivalently

$$\left| U\left(f, -\frac{1}{\alpha}; z\right) - 1 \right| < \lambda, \quad z \in \mathbb{D}.$$

Additionally, if $z/f(z) \neq 0$ for all $z \in \mathbb{D}$, then $f \in \mathcal{U}(\lambda, -1/\alpha)$. Both implications are sharp, i.e., the coefficient λ can not be replaced by a smaller number so that the corresponding implications hold.

Proof. For $A = (1+\alpha)\lambda$ and $B = \lambda$, the condition $0 < \alpha \leq \frac{1}{\lambda} - 1$ ensures $\lambda < (1+\alpha)\lambda \leq 1$, i.e. $-1 \leq B < A \leq 1$. Further, $B = \lambda \neq 0$, $c_1 = \frac{A-B}{\alpha B} = 1$ and

$$1 + \alpha - \frac{\alpha}{1 + \lambda z} = \frac{1 + (1+\alpha)\lambda z}{1 + \lambda z} = \frac{1 + Az}{1 + Bz}.$$

Thus, all conditions of Theorem 1 hold which implies the conclusions of this corollary. ☐

Choosing some exact values of A and/or B in Corollary 1, we directly receive

Corollary 3. *Let $f \in \mathcal{A}$, $-1 \leq B < A \leq 1$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Also, let $f'(z) \neq 0$ and $z/f(z) \neq 0$ for all $z \in \mathbb{D}$.*

(i) If $\alpha \leq -2$ and $\operatorname{Re} J(f, \alpha; z) > 0$ $(z \in \mathbb{D})$, *then*

$$f \in \mathcal{U}(2^{-2/\alpha} - 1, -1/\alpha).$$

($A = 1$ and $B = -1$ in Corollary 1);

(ii) *If* $0 \le \beta < 1$, $\alpha < 0$, $2(1-\beta) \le |\alpha|$ *and* $\mathrm{Re} J(f,\alpha;z) > \beta$ $(z \in \mathbb{D})$, *then*

$$f \in \mathcal{U}(2^c - 1, -1/\alpha). \quad (A = 1 - 2\beta \text{ and } B = -1)$$

where $c = -\frac{2(1-\beta)}{\alpha}$. *(*$A = 1-2\beta$ *and* $B = -1$ *in Corollary 1);*

(iii) *If* $0 < A \le 1$, $A \le |\alpha|$ *and* $|J(f,\alpha;z) - 1| < A$ $(z \in \mathbb{D})$, *then*

$$f \in \mathcal{U}(e^c - 1, -1/\alpha). \quad (B = 0)$$

where $c = \frac{A}{\alpha}$ *(*$B = 0$ *in Corollary 1).*

3. Examples

In this section we give two examples where using Corollaries 1 and 2 we will show that a given function is both, of α-convex Janowski type and in the class $\mathcal{U}$. A direct conclusion (by the definitions of the classes) is more delicate.

Example 1. Let $\alpha > 0$, $0 < \lambda \le \frac{1}{1+\alpha}$ and $a \in \mathbb{R}$. Also, let $|a| < 1$,

$$\frac{|a|}{\alpha} + \frac{1}{1-|a|} \le \frac{1}{1+\lambda} \tag{6}$$

and $f(z) = z \cdot e^{az}$. Then

$$f(z) \in \mathcal{M}[(1+\alpha)\lambda, \lambda, \alpha] \cap \mathcal{U}(\lambda, -1/\alpha).$$

Proof. It is easy to verify that $f \in \mathcal{A}$, $f'(z) \neq 0$ and $z/f(z) \neq 0$ for all $z \in \mathbb{D}$. Also,

$$J(f,\alpha;z) \equiv (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + \alpha + az - \frac{\alpha}{1+az}.$$

Further,

$$J(f,\alpha;z) \prec 1 + \alpha - \frac{\alpha}{1+\lambda z} \quad \Leftrightarrow \quad az - \frac{\alpha}{1+az} \prec -\frac{\alpha}{1+\lambda z}$$

and the last subordination is equivalent to

$$\sup_{|z|=1}\left|az - \frac{\alpha}{1+az}\right| \le \inf_{|z|=1}\left|-\frac{\alpha}{1+\lambda z}\right| = \frac{\alpha}{1+\lambda}. \tag{7}$$

Now, let put $z = e^{i\theta}$, $t = \cos\theta$ and define a function

$$h(t) \equiv \left| az - \frac{\alpha}{1+az} \right|^2 \Bigg|_{z=e^{i\theta}} = a^2(1+\alpha) - 2\alpha t + \frac{\alpha[\alpha + a^2(1-a^2)]}{1+2at+a^2}.$$

Next,

$$h'(t) = -2a\alpha \cdot \left[1 + \frac{\alpha + a^2(1-a^2)}{(1+2at+a^2)^2} \right]$$

has constant sign, i.e.,

$$\sup_{|t|\le 1} h(t) = \max\{h(-1), h(1)\} = \left\{ \begin{array}{ll} h(-1), & a \ge 0 \\ h(1), & a < 0 \end{array} \right\} = \frac{[\alpha + |a|(1-|a|)]^2}{(1-|a|)^2}.$$

Thus, inequality (7) holds if and only if

$$\frac{\alpha + |a|(1-|a|)}{1-|a|} \le \frac{\alpha}{1+\lambda}$$

which is equivalent to (6). This means that $f(z) \in \mathcal{M}[(1+\alpha)\lambda, \lambda, \alpha]$. Finally, applying Corollary 2 we receive $f(z) \in \mathcal{U}(\lambda, -1/\alpha)$. □

Example 2. Let $\alpha > 0$, $a > 0$ and $0 < \gamma \le \frac{1}{1+a}$. Then

$$f(z) = z \cdot \left(1 + \frac{\gamma}{1+a} \cdot z \right)^a \in \mathcal{M}[(1+\alpha)\lambda, \lambda, \alpha] \cap \mathcal{U}(\lambda, -1/\alpha).$$

Proof. It is easy to check that $f \in \mathcal{A}$, $f'(z) \ne 0$ and $z/f(z) \ne 0$ for all $z \in \mathbb{D}$. Also,

$$J(f,\alpha;z) \equiv (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + \frac{a\gamma z}{1+\gamma z} = \frac{1+\gamma(1+a)z}{1+\gamma z},$$

i.e., $f(z) \in \mathcal{M}[A, B, \alpha]$ for $A = \gamma(1+a)$ and $B = \gamma$. Also,

$$f'(z) = f(z) \cdot (1+a) \cdot \left[1 - \frac{a}{1+a+\gamma z} \right]$$

and

$$\frac{z}{f(z)} = \left(1 + \frac{\gamma}{1+a} \cdot z \right)^{-a}$$

do not vanish on the unit disk. The rest follows directly form Corollary 1. □

References

[1] L.A. Aksentév, Sufficient conditions for univalence of regular functions (Russian), *Izv. Vysš. Učebn. Zaved. Matematika*, **4**, No 3 (1958), 3–7.

[2] P.L. Duren, *Univalent Functions*, Springer-Verlag (1983).

[3] R. Fournier, S. Ponnusamy, A class of locally univalent functions defined by a differential inequality, *Complex Var. Elliptic Equ.* **52**, No 1 (2007), 1–8.

[4] W. Janowski, Some extremal problems for certain families of analytic functions I, *Annales Polonici Mathematici*, **28** (1973), 297–326.

[5] S.S. Miller and P.T. Mocanu, *Differential Subordinations, Theory and Applications*, Marcel Dekker, New York-Basel (2000).

[6] S.S. Miller and P.T. Mocanu, Differential subordinations and univalent functions, *Michigan Math. J.* **28**, No 2 (1981), 157–172.

[7] S.S. Miller and P.T. Mocanu, On some classes of first-order differential subordinations, *Michigan Math. J.* **32**, No 2 (1985), 185–195.

[8] M. Nunokawa, M. Obradovic, and S. Owa, One criterion for univalency, *Proc. Amer. Math. Soc.* **106** (1989), 1035–1037.

[9] M. Obradovic and S. Ponnusamy, Product of univalent functions, *Math. Comput. Modelling* **57**, No 3-4 (2013), 793-799.

[10] M. Obradovic and S. Ponnusamy, Criteria for univalent functions in the unit disk, *Arch. Math. (Basel)* **100**, No 2 (2013), 149-157.

[11] M. Obradovic and S. Ponnusamy, On harmonic combination of univalent functions, *Bull. Belg. Math. Soc. Simon Stevin* **19**, No 3 (2012), 461-472.

[12] M. Obradovic and S. Ponnusamy, New criteria and distortion theorems for univalent functions, *Complex Variables Theory Appl.* **44** (2001), 173-191.

[13] M. Obradovic, S. Ponnusamy, V. Singh and P. Vasundhra, Univalency, starlikeness and convexity applied to certain classes of rational functions, *Analysis (Munich)* **22**, No 3 (2002), 225-242.

[14] M. Obradovic and S. Ponnusamy, Univalence and starlikeness of certain integral transforms defined by convolution of analytic functions, *J. Math. Anal. Appl.* **336** (2007), 758-767.

[15] M. Obradovic and S. Ponnusamy, Coefficient characterization for certain classes of univalent functions, *Bull. Belg. Math. Soc. (Simon Stevin)* **16** (2009), 251-263.

[16] S. Ozaki, M. Nunokawa, The Schwarzian derivative and univalent functions, *Proc. Amer. Math. Soc.* **33** (1972), 392–394.