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DIAGONAL MATRIX SCALING AND H-MATRICES

Mohammad H. Koulaei

Department of Mathematics
Faculty of Sciences
University of Zanjan
Zanjan 45195-313, IRAN

Abstract: The iterative methods for characterization of H-matrices consider the problem of finding a positive diagonal matrix D such that AD is strictly diagonally dominant. In this paper we consider this property and use the Gordan's theorem of the alternative to find a linear feasibility problem which can be solved efficiently by pivoting methods and gives us a criterion for deciding about the H-character of a given matrix. We also describe matrix scaling problem and show that there is a matrix corresponding to any given matrix A such that its scalability is equivalent to the H-character of A.

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Key Words: H-matrix, generalized strictly diagonally dominant, linear feasibility problem, diagonal matrix scaling

1. Introduction

The class of H-matrices [4, 9] appears in numerical analysis and some applications such as the linear complementary problem [1]. So, the characterization of H-matrices becomes a subject of some researches. Finding equivalent conditions

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of H-matrices which are of practical use is important for this characterization. There are different ways to define an H-matrix. The most common one is based on (row-wise) strictly diagonally dominant which is defined as follows.

Definition 1.1. A matrix $X \in \mathbb{R}^{n\times n}$ is said to be (row-wise) strictly diagonally dominant matrix if and only if

$$|x_{ii}| > \sum_{j=1, j\neq i}^{n} |x_{ij}|, \quad i = 1(1)n.$$

Now we can consider following definition for an H-matrix.

Definition 1.2. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be an H-matrix if and only if there exists a diagonal matrix D with positive diagonal elements, such that AD is row-wise strictly diagonally dominant.

Some iterative algorithms have been proposed based on this definition to find out if a matrix is an H-matrix or not, [2, 3, 12]. They all try to find a point diagonal matrix D to show that Definition 1.2 holds.

In this paper we are going to use the Gordan's theorem of alternative [5] to show that for a given matrix $A \in \mathbb{R}^{n\times n}$, being an H-matrix is equivalent to existing a solution for a system of linear inequality. There are some efficient algorithms [6] that can be used for finding the solution of a set of linear inequality if there exists any, or confirming that the system is inconsistent.

Definition 1.2 turns to be a problem of finding an interior point of a polyhedron. In [11] Jin and Kalantari used the idea of matrix scaling in [10] for finding the interior point of a polyhedron or showing that a system of linear strict inequality has no solution. We use the similar idea and give a relationship between the scalability and H- character of a matrix. Then we use the Khachyan-Kalantari algorithm [10, 7] for finding suitable D for which AD is strictly diagonally dominant or showing that the polyhedron corresponding to Definition 2.1 does not have any interior point.

This paper is organized as follows. In Section 2, we introduce the linear feasibility problem. We discuss its relation to H-matrices. We state how we can use the existing algorithms for solving linear feasibility problem to check if a given matrix is an H-matrix or not. In Section 3, we describe the diagonal matrix scaling problem and a path-following algorithm for solving this problem. We prove some results on the relationship between diagonal matrix scaling problem and characterisation of H-matrices. Some numerical result for testing

the new ideas are presented in Section 4. Finally, conclusion and final remarks are given in Section 5.

2. Feasiblility and H-Matrices

In this section, we first restate the definition of an H-matrix with another notation. Next we describe how to turn into linear feasibility problem to decide about the H-character of a matrix.

Definition 2.1. The comparison matrix of $A \in \mathbb{R}^{n\times n}$ is the matrix $\mathcal{M}(A) = [m_{ij}]$ with elements

$$m_{ij=}\begin{cases}|a_{ii}|, & i = j = 1(1)n,\\ -|a_{ij}|, & i \neq j = 1(1)n.\end{cases}$$

Definition 2.2. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be (row-wise) generalized strictly diagonally dominant if and only if there exist positive diagonal matrix D such that AD is strictly diagonally dominant.

Therefore a matrix $A \in \mathbb{R}^{n\times n}$ is an H-matrix if and only if its comparison matrix is generalized strictly diagonally dominant. That means the definition of H-matrix is equivalent to the existence of a positive vector d such that $\mathcal{M}(A)d > 0$. Consider the following system:

$$\begin{aligned}&\mathcal{M}(A)d > 0,\\ &d > 0, \quad d \in \mathbb{R}^n,\end{aligned} \tag{1}$$

then A is an H-matrix if and only if system (1) is feasible. Feasibility of the system (1) can be checked through finding a nonnegative nonzero solution of a system of homogeneous linear equations. Following theorem shows this relation.

Theorem 2.3. *[5] (Gordan's theorem) Let G be a real $m \times n$ matrix, then exactly one of the following two systems has a solution:*

System 1: $Gx > 0$ for some $x \in \mathbb{R}^n$

System 2: $G^T y = 0,\ y \geq 0$ for some nonzero $y \in \mathbb{R}^m$.

By applying Gordan's theorem to the system (1) we obtain following corollary which gives the linear feasibility problem that can be used for characterization of H-matrices.

Corollary 2.4. *Let $\mathcal{M}(A)$ be the comparison matrix of $A \in \mathbb{R}^{n\times n}$, then exactly one of the following two systems has a solution:*

System 1: $\mathcal{M}(A)x > 0,\ x > 0$ for some $x \in \mathbb{R}^n$

System 2: $\mathcal{M}(A)^T w + v = 0,\ (w^T, v^T) \geq 0$ for some nonzero (w^T, v^T) with $w, v \in \mathbb{R}^n$.

Let $\mathcal{M}(A)$ be the comparison matrix of $A \in \mathbb{R}^{n\times n}$. Consider the system

$$\begin{aligned} &Bz = 0, \\ &z \geq 0,\ z \neq 0,\ z \in \mathbb{R}^{2n}, \end{aligned} \tag{2}$$

in which $B = (\mathcal{M}(A)^T, I)$ and $z^T = (w^T, v^T)$. Then, system (1) is feasible if and only if system (2) is infeasible. That means A is an H-matrix if and only if system (2) is infeasible.

For checking the feasibility of system (2) one can consider a linear programming problem with feasible set similar to the system (2) and appropriate objective function and then applying simplex algorithm [6] for the resulting problem. Since the coefficient matrix in system (2) is of order $n \times 2n$, in solving the corresponding LP, it is needed $\mathcal{O}(n^2)$ computations in the worst case, at each iteration of the simplex algorithm. This complexity is the same as some iterative methods that proposed for characterization of H-matrices [2, 12]. Those algorithms need to update a matrix of order $n\times n$ at each iteration and therefore the amount of computation that they need for one iteration is $\mathcal{O}(n^2)$.

3. Diagonal Matrix Scaling

In this section we introduce the diagonal matrix scaling problem. We can use this problem for testing the feasibility of system (2) and as a result this gives a new criterion for characterization of H-matrices.

Following definition of positive semidefinite matrix is used for defining the scaling problem.

Definition 3.1. A matrix $Q \in \mathbb{R}^{n\times n}$ is called positive semidefinite if $x^T Q x \geq 0$ for all vectors $x \in \mathbb{R}^n$.

Given an $n \times n$ symmetric positive semidefinite matrix Q, either find a positive diagonal matrix X which scales Q into a doubly quasi-stochastic matrix

$$XQXe = e, \quad X = \operatorname{diag}(x_1, x_2, \cdots, x_n) > 0,$$

or prove that Q is not scalable. Here $e = (1, \cdots, 1)^T \in \mathbb{R}^n$.

Let $x = (x_1, x_2, \cdots, x_n)^T$, $x^{-1} = (1/x_1, 1/x_2, \cdots, 1/x_n)^T$, then this problem can be written as

$$Qx - x^{-1} = 0, \quad x > 0. \tag{3}$$

In [10] Khachyan and Kalantari proposed a path following algorithm for solving the scaling problem (3). Based on this algorithm and following theorem Chvatal [7] suggested a procedure for solving linear programming as homogeneous problem.

Theorem 3.2. *[7] Every symmetric positive semidefinite matrix Q has precisely one of the following two properties:*
(i) There is a diagonal matrix D such that $De > 0$ and $(DQD)e = e$,
(ii) There is a nonnegative nonzero vector x such that $Qx = 0$.

The Khachyan-Kalantari algorithm can be summarized as follows [10, 7].

Algorithm 1: Khachyan-Kalantari algorithm

input:
Symmetric positive semidefinite $n \times n$ matrix Q;
positive number δ, ϵ less than 1;
Set $\rho = (1 - 1/(1 + 4\sqrt{n}))^{1/2}$;
Set $D_0 = I$, $k = 0$;
while $(D_k e)^T Q(D_k e) \geq \epsilon \|D_k e\|^2$ *and* $\|D_k Q D_k e - e\| \geq 3/4$ **do**
  Solve the system $(I + D_k Q D_k)r = e - D_k Q D_k e - \rho^k D_k(e - Qe)$;
  Set $D_{k+1} = \rho \mathrm{diag}(D_k(e + r))$, $k = k + 1$;
if $(D_k e)^T Q(D_k e) \leq \epsilon \|D_k e\|^2$ **then**
  Return the vector $\|D_k e\|^{-1} D_k e$;
else
  $D_0 = D_k$, $k = 0$;
  while $\|D_k Q D_k e - e\| \geq \delta$ **do**
    Solve the system $(I + D_k Q D_k)r = e - D_k Q D_k e$;
    Set $D_{k+1} = \mathrm{diag}(D_k(e + r))$, $k = k + 1$;
  Return the matrix D_k

The following theorem uses Theorem 3.2 and proposes a relationship between H-matrices and scalability.

Theorem 3.3. *Let $A \in \mathbb{R}^{n\times n}$ and $B = (\mathcal{M}(A)^T, I)$. Then A is an H-matrix if and only if B^TB, is scalable.*

Proof. Assume that A is an H-matrix. From Corollary 2.4, system (2) is infeasible. Therefore system $B^TBz = 0,\ 0 \neq z \geq 0$ has no solution, since otherwise $\|Bz\|^2 = z^TB^TBz = 0$ which contradicts the infeasibility of system (2). Thus it follows that the case (i) of Theorem 3.2 holds and therefore B^TB is scalable.

Conversely, if A is not an H-matrix, then system (2) is feasible. Thus the case (ii) of Theorem 3.2 holds and therefore B^TB is not scalable. □

Khachyan-Kalantari's algorithm can be used for finding positive diagonal matrix D for which the matrix Q is scalable or concluding that the system $Qx = 0$, has nonnegative nonzero solution. Theorem 3.3 shows that this algorithm can be used for finding the positive diagonal matrix for which Definition 1.2 holds. The following theorem shows how this can be done by the help of diagonal matrix scaling.

Theorem 3.4. *Let $A \in \mathbb{R}^{n\times n}$ and $B = (\mathcal{M}(A)^T, I)$. Then following conditions are equivalent:*
(i) A is an H-matrix.
(ii) B^TB is scalable.
(iii) There exists $(w^T, v^T) > 0$ such that $\mathcal{M}(A)v^{-1} = w^{-1}$.

Proof. It follows from Theorem 3.3 that (i) implies (ii).

Let B^TB be scalable, so there is positive diagonal matrix Z such that $ZB^TBZ = e$. Let $z = \mathrm{diag}(Z)$ and $z^T = (w^T, v^T)$, then we have $B^TBz = z^{-1}$. Since $B = (\mathcal{M}(A)^T, I)$, it is concluded that

$$\begin{bmatrix} \mathcal{M}(A)\mathcal{M}(A)^T & \mathcal{M}(A) \\ \mathcal{M}(A)^T & I \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} w^{-1} \\ v^{-1} \end{bmatrix},$$

then we have

$$\mathcal{M}(A)(\mathcal{M}(A)^Tw + v) = w^{-1}$$
$$\mathcal{M}(A)^Tw + v = v^{-1}.$$

By using $\mathcal{M}(A)^Tw + v$ instead of v^{-1} in the first equality, one has

$$\mathcal{M}(A)v^{-1} = w^{-1}, \tag{4}$$

which proves that (ii) implies (iii).

Finally suppose that there exists $(w^T, v^T) > 0$ such that $\mathcal{M}(A)v^{-1} = w^{-1}$. It is concluded that $d = v^{-1}$ is a feasible solution for system (1). So A is (row-wise) generalized strictly diagonally dominant matrix and therefore H-matrix. Thus (iii) implies (i). □

The algorithmic implication of Theorem 3.4 is as follows. In order to find if A is an H-matrix or non H-matrix we test if B^TB, $B = (\mathcal{M}(A)^T, I)$, is scalable by applying the Khachyan-Kalantari diagonal matrix scaling algorithm (Algorithm 1). Either the algorithm exclusively computes positive diagonal matrix Z such that $B^TBZe = Z^{-1}e$ is approximately satisfied which in turn implies that A is an H-matrix, or it gives $0 \neq z \geq 0$ such that $Bz = 0$ approximately which implies A is not an H-matrix. In the first case, it is concluded from (4) that such Z gives suitable $d > 0$ with $\mathcal{M}(A)d > 0$.

From computational point of view Algorithm 1 needs to solve a linear system at each iteration in both while loops, that means at each iteration it costs $\mathcal{O}(n^3)$ computations. It is not as good as $\mathcal{O}(n^2)$ computational complexity of the previous section.

4. Numerical Examples

In this section we present a set of examples that have been chosen from different references. The algorithms are coded in MATLAB 7.

In Section 2 it was mentioned that LP can be used as a tool for solving a linear feasibility problem. We consider following LP for testing the feasibility of system (2):

$$\begin{array}{ll} \max & e^Tw \\ s.t. & \mathcal{M}(A)w \leq 0, \\ & w \geq 0. \end{array} \tag{5}$$

If problem (5) has optimal solution with zero objective function it is concluded that system (2) is infeasible, which means A is an H-matrix. Otherwise A should be non H-matrix. For solving this LP problem we used `linprog` function of MATLAB with default options for all parameters except the solver that was Simplex algorithm.

For implementing Algorithm 1, the parameters were chosen as $\epsilon = 10^{-10}$ and $\delta = 10^{-10}$. The code uses MATLAB's function `pcg` to solve the linear system inside the first and second while loop of the algorithm.

Here are the examples.

Example 4.1. Consider a class of matrices $A(a_{12}) \in \mathbb{R}^{5\times 5}$,

$$A(a_{12}) = \begin{bmatrix} -1 & a_{12} & 0 & 0 & 0 \\ 0.5 & -1 & 0 & -0.6 & 0 \\ 0 & -0.1 & 1 & 0 & 0.5 \\ 0 & 0.5 & 0 & 1 & -0.5 \\ -0.2 & 0.1 & 0.3 & 0 & -1 \end{bmatrix},$$

where $a_{12} \in (1, 2)$ [8]. For the choice of $a_{12} = 1.146391$, Algorithm 1 gave the positive vector

$$d^T = [1.850428243935277, 1.614133445708352, 0.502547652077170, \\ 1.148198872568836, 0.682263895836364]$$

that satisfies system $\mathcal{M}(A(a_{12}))d > 0$. It is concluded that for $a_{12} = 1.146391$, $A(a_{12})$ is an H-matrix.

In [8] another choice for a_{12} has been made to somehow determine computationally a "small" interval such that both cases of H- and non H-matrix happen for $A(a_{12})$. So $a_{12} = 1.416392$ was considered for this purpose. Algorithm 1 gave the vector

$$z^T = [0.422794153595375, 0.740978072703923, 0.078442941614187, \\ 0.444585910334816, 0.261508455798654, 0.000003086842709, \\ 0.000003538725325, 0.000011364246659, 0.000004974657872, \\ 0.000008371652444]$$

as a solution of system (2) which means that $A(a_{12})$ is not H-matrix for $a_{12} = 1.146392$.

Example 4.2. The following example has been chosen from [3]:

$$A = \begin{bmatrix} 1 & 0.001 & 0 & 0 & 0 & 0 & 0.03 \\ 0.02 & 1 & 0 & 0 & 0 & 0.01 & 0 \\ 0 & 0 & 1 & 0 & 0.1 & 0.03 & 0.01 \\ 0 & 0 & 20 & 1 & 0 & 0.05 & 0.01 \\ 0 & 0 & 0 & 4 & 1 & 0.03 & 0.02 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The matrix A is not an H-matrix, since Algorithm 1 gave the vector

$$z^T = [0.000003537315821, 0.000003570715846, 0.270927878578392,$$
$$0.026137349256237, 0.018574879401247, 0.119034914434428,$$
$$0.149606423598557, 0.000003533745122, 0.000003499985449,$$
$$0.251819104378512, 0.048162168503303, 0.008517911457066,$$
$$0.845219294111824, 0.329875489832633]$$

that is the solution of system (2).

The same characterizations obtained for all examples by solving the corresponding linear programming problem (5) by MATLAB function `linprog`. The algorithm converged in 5 and 4 iterations respectively for two cases of Example 1, and 3 iterations for Example 2.

5. Conclusion

In this paper, we have discussed how H-matrices can be related to the linear feasibility problem and found a way to characterize H-matrices by using the available algorithms for solving this problem. We also proved that there is a matrix corresponding to any matrix $A \in \mathbb{R}^{n\times n}$ whose scalability is equivalent to the H-matrix character of A. Then we applied Khachyan-Kalantari algorithm to find either a positive vector d such that $\mathcal{M}(A)d > 0$ and therefore confirm that A is an H-matrix, or find a nonnegative nonzero vector z such that $z^T B^T Bz = 0$, $(B = (\mathcal{M}(A)^T, I))$, and concluding that A is not an H-matrix.

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