International Journal of Applied Mathematics

Volume 28 No. 5 2015, 495-505
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v28i5.3

ANALYSIS OF DYNAMICAL BEHAVIORS FOR A DELAYED SIS EPIDEMIC MODEL WITH INCUBATION PERIOD

Qiming Liu[1 §], Changsong Deng[2]
[1,2]Institute of Applied Mathematics
Shijiazhuang Mechanical Engineering College
Shijiazhuang, 050003, P.R. CHINA

Abstract: In this paper, we present a new epidemic spreading SIS model with time delays on scale-free networks. We give formula of the basic reproductive number R_0 for the model. The global asymptotic stability of a disease-free equilibrium when $R_0 < 1$ and the uniformly persistent of disease when $R_0 > 1$ are proved. Numerical simulations are given to demonstrate the main results.

AMS Subject Classification: 97M50, 97P70, 34K20, 34K12
Key Words: scale-free networks, SIS epidemic model, stability, time delay

1. Introduction and Model Formulation

In epidemiology, mathematical models are the basic conceptual tools used to study epidemic dynamics and to formulate control strategies. In recent years, motivated by the seminal works on the scale-free (SF) property [1], the studies of epidemic spreading on SF networks have attracted extensive interests within the academic community [2]–[11]. A SF network is a network whose degree distribution follows a power law, at least asymptotically. That is, the probability that a node chosen randomly from the network having degree k follows a power law distribution $P(k) = ck^{-r}$, where γ is the power-law exponent whose value is typically in the range $2 \leq \gamma < 3, c = 1/\sum k^{-\gamma}$.

It is well known that time delay can play an important role in the epidemic model. It can be introduced to represent the infection periods of infective

Received: June 6, 2015

§Correspondence author

members, the periods of recovered individuals with immunity and the periods of quarantined members. Although there are many studies of the dynamical behaviors of the epidemic models with time delays on homogeneous networks [12], the epidemic models with time delay on the spreading behaviors in SF networks attract a few attentions. In 2013, Xia et al. [13] investigate the effects of delaying the time to recovery by a set of ordinary differential equations. Recently, Liu et al. [14] and Wang et al. [15] discussed the effect of infection delay for SIS model and SIR model on SF networks respectively.

In paper [16], authors consider an epidemic SIS model with nonlinear infectivity, as well as birth and death of nodes and edges on SF networks. Based on the classical SIS model, the dynamical mean-field reaction rate equation is established as follows:

$$\begin{cases} \dfrac{dS_k(t)}{dt} = b(S_k(t) + I_k(t)) - dS_k(t) - \lambda(k)S_k(t)\Theta(t) + aI_k(t), \\ \dfrac{dI_k(t)}{dt} = \lambda(k)S_k(t)\Theta(t) - (a+d)I_k(t), \end{cases} \tag{1}$$

where $S_k(t)$ and $I_k(t)$ represent the relative densities of the susceptible and infected nodes; the natural births and deaths are proportional to the node size with birth rate b and death rate d; a is the recovery rate; and the factor $\Theta(t)$ represents the probability that any given link points to an infected site,

$$\Theta(t) = \frac{\sum\limits_k \varphi(k)p(k)I_k(t)}{\langle k \rangle}, \tag{2}$$

where $\lambda(k) > 0$ is the correlated (k-dependent) infection rate and $\varphi(k)$ means the occupied edges which can transmit the disease, which can take various forms, such as $\lambda(k) = k\lambda$(where λ is the spreading rate at which each susceptible individual acquires the infection from an infected neighbor during one time step) and $\varphi(k) = k$ in [2], [3], $\lambda(k) = \lambda k A(k)$ and $\varphi(k) = kT(k)$ (where A(k) is the probability that a susceptible node would actually admit an infection through an edge connected to the infected node, similar to T(k)) in [6], $\lambda(k) = k\lambda$ and $\varphi(k) = ak^\alpha/(1 + bk^\alpha)$ in [4], $\lambda(k) = k\lambda$ and $\varphi(k) = A$ in [5]. Note that $\varphi(k)$ in [4]-[6] is suitable than $\varphi(k) = k$ in [2], [3] since an infected cannot contact all acquaintances in one time step.

Based on the the work of reference [16], we consider a new epidemic SIS model with time delay on SF networks to investigate the impact of incubation delay. By applying the mean-field technique, the dynamics model based on delay differential equations when $t > \tau$ is established as follows:

$$\begin{cases} \dfrac{dS_k(t)}{dt} = b(S_k(t) + I_k(t)) - dS_k(t) - \lambda(k)S_k(t)\Theta(t-\tau) + aI_k(t), \\ \dfrac{dI_k(t)}{dt} = \lambda(k)S_k(t)\Theta(t-\tau) - (a+d)I_k(t), \end{cases} \tag{3}$$

where τ is a kind of incubation period of the computer virus or the disease of population, which means that the infected code begin to infect the other individuals after τ. Other parameters have the same meanings as in (1), and all the parameters are positive.

The initial condition of system (3) is

$$S_k(\theta) = \varphi_k(\theta), I_k(\theta) = \phi_k(\theta), \theta \in [-\tau, 0], \tag{4}$$

where $\omega_k = (\varphi_k(\theta), \phi_k(\theta)) \in C$ are nonnegative continuous on $[-\tau, 0]$ and $\varphi_k(0) > 0, \phi_k > 0$ for $\theta = 0$. C denote the Banach space $C([-\tau, 0], R^{2n})$, with the norm $\|\omega\| = \sup_{-h \le \theta \le 0} |\omega|$, and $|\omega|$ is Euclidean norm of R^{2n}.

We suppose that the total number of nodes is constant, that is, deaths are balanced by births, hence $b = d$ and $S_k(t) + I_k(t) \equiv 1$. This is a suitable assumption since the adding and removal nodes and edges only take a small proportion in a sufficiently large network such as social networks and Internet. Thus, system (3) becomes the following model:

$$\frac{dI_k(t)}{dt} = \lambda(k)(1 - I_k(t))\Theta(t-\tau) - (a+d)I_k(t). \tag{5}$$

In this paper, the global behaviors of the model (5) with initial condition (4) are studied mathematically. First, the formula of the basic reproductive number is given. Next the dynamical behaviours of the system are analyzed. Finally numerical simulations are given to demonstrate the main results.

2. Analysis of the Epidemic Model

In this section, we analyze the model (5) theoretically.

Denote

$$R_0 = \frac{\langle \lambda(k)\varphi(k)\rangle}{(a+b)\langle k\rangle}, \qquad I(t) = \sum_k P(k)I_k(t). \tag{6}$$

Theorem 1. *There is always a virus-free equilibrium* $E_0\{0, ..., 0\}$ *and when* $R_0 > 1$, *then system (5) has a unique endemic equilibrium* E_*.

Proof. It is easily verified that E_0 is always an equilibrium of system (5). Denote $I_k = I_k^*$ (some constants) and substitute them into (5), we have

$$\lambda(k)(1 - I_k^*)\Theta^* - (a + b)I_k^* = 0,$$

where

$$\Theta^* = \frac{1}{\langle k\rangle}\sum_i \varphi(i)p(i)I_k^*. \tag{7}$$

This yields that

$$I_k^* = \frac{\lambda(k)\Theta^*}{(a + b) + \lambda(k)\Theta^*}. \tag{8}$$

Substituting (8) into (2), we obtain the self-consistency equality

$$\Theta^* = \frac{1}{\langle k\rangle}\sum_k \frac{\lambda(k)\varphi(k)P(k)\Theta^*}{(a + b) + \lambda(k)\Theta^*} = f(\Theta^*). \tag{9}$$

Obviously, $\Theta^* = 0$ always satisfies (9), it follows that from (8) that the disease-free equilibrium E_0 of system (5) always exists. Note that

$$\left.\frac{df(\Theta^*)}{d\Theta^*}\right|_{\Theta^*=0} = \frac{\langle\lambda(k)\varphi(k)\rangle}{(a + b)\langle k\rangle} = R_0,$$

and

$$\frac{df^2(\Theta^*)}{d\Theta^{*2}} = \frac{1}{\langle k\rangle}\sum_k \varphi(k)p(k)\frac{-2\lambda(k)^2}{((a + b) + \lambda(k)\Theta^*)^3} < 0.$$

Obviously, if $R_0 > 1$, the equation (9) has a unique positive solution, consequently, system (5) has a unique positive equilibrium $E_*(I_1^*, I_2^*, \cdots, I_n^*)$ since (8) holds, where n is the maximum number of contact each individual.

Lemma 1. ([17]) *Consider the following equation:*

$$\dot{x} = a_1 x(t - \tau) - a_2 x(t),$$

where $a_1, a_2, \tau > 0$*;* $x(t) > 0$ *for* $-\tau \le t \le 0$*. We have:*

(i) *if* $a_1 < a_2$*, then* $\lim_{t\to+\infty} x(t) = 0$,

(ii) *if* $a_1 > a_2$*, then* $\lim_{t\to+\infty} x(t) = +\infty$.

Lemma 2. ([18, p.273-280]) *Let* X *be a complete metric space,* $X = X^0 \cup \partial X^0$*, where* ∂X^0*, assumed to be nonempty, is the boundary of* X^0*. Assume the* C^0*-semigroup* $T(t)$ *on* X *satisfies* $T(x) : X^0 \to X^0, T(x) : \partial X^0 \to \partial X^0$ *and:*

(i) *there is a* t_0 *such that* $T(t)$ *is compact for* $t > t_0$*; and*
(ii) $T(t)$ *is point dissipative in* X*; and*
(iii) $\tilde{A}_\partial$ *is isolated and has an acyclic covering* M*.*

Then $T(t)$ *is uniformly persistent if and only if, for each* $M_i \in M$*,*

$$W^s(M_i) \cap X^0 = \emptyset,$$

where $\tilde{A}_\partial = \bigcup_{x\in A_\partial} \omega_x$*, and* $\omega(x)$ *is the omega limit set of* $T(x)$ *through* x*,* A_∂ *is global attractor of* $T_\partial(t)$ *in* ∂X^0 *in which* $T_\partial(t) = T(t)\mid_{\partial X^0}$*.*

Theorem 2. *The virus-free equilibrium* E_0 *is unstable when* $R_0 > 1$ *and global asymptotic stable when* $R_0 < 1$*. When* $R_0 > 1$*, the disease is uniformly persistent, i.e., there exists a positive constant* ε*, such that* $\liminf_{t\to\infty} I(t) > \varepsilon$*.*

Proof. First, we discuss the stability of equilibrium E_0 of system (5).

Let us consider the Lyapunov function $V(t)$

$$V(t) = \frac{1}{2}\Theta^2(t) + \mu\int_{t-\tau}^{t}\Theta^2(s)ds.$$

Calculating the derivative of $V(t)$ along the solution of system (5), we obtain

$$\begin{aligned} V'(t) &= \Theta(t)[\frac{1}{\langle k\rangle}\sum_k \varphi(k)P(k)(\lambda(k)(1-I_k(t))\Theta(t-\tau)-(a+b)I_k(t))] \\ &\quad +\mu(\Theta^2(t)-\Theta^2(t-\tau)), \\ &\leq \frac{\lambda(k)\varphi(k)}{\langle k\rangle}\Theta(t)\Theta(t-\tau)-(a+b)\Theta^2(t)+\mu\Theta^2(t)-\mu\Theta^2(t-\tau), \\ &\leq \frac{1}{2}[\Theta^2(t)+\Theta^2(t-\tau)]\frac{\langle\lambda(k)\varphi(k)\rangle}{\langle k\rangle} \\ &\quad -(a+b)\Theta^2(t)+\mu\Theta^2(t)-\mu\Theta^2(t-\tau). \end{aligned}$$

Let $\mu = \frac{1}{2}\frac{\langle\lambda(k)\varphi(k)\rangle}{\langle k\rangle}$, we have from (2) that

$$V'(t) \leq [\frac{\langle\lambda(k)\varphi(k)\rangle}{\langle k\rangle}-(a+b)]\Theta^2(t).$$

Note that $\langle\lambda(k)\varphi(k)\rangle/\langle k\rangle-(a+b) < 0$, i.e., $R_0 < 1$, hence $V'(t) \leq 0$. Therefore, E_0 is globally asymptotically stable if $R_0 < 1$.

If $R_0 > 1$, we will proof that the equilibrium E_0 of system (5) is unstable.

The linear system of (5) is

$$\frac{dI_k(t)}{dt} = \lambda(k)\langle k\rangle^{-1}\sum_k \varphi(k)P(k)I_k(t-\tau)-(a+b)I_k(t). \tag{10}$$

Assuming ρ is eigenvalue of characteristic equation of system (5), the Jacobian matrix of system (5) at the equilibrium E_0 as follows:

$$A = (a_{ij})_{n\times n},$$

where

$$a_{ij} = \begin{cases} \dfrac{\lambda(i)\varphi(j)P(j)\mathrm{e}^{-\rho\tau}}{\langle k\rangle}, & i \neq j \\ \dfrac{\lambda(i)\varphi(j)P(j)\mathrm{e}^{-\rho\tau}}{\langle k\rangle} - (a+b), & i = j. \end{cases}$$

For matrix A, the first row multiplied by $-\lambda(m)/\lambda(1)$ add to the mth row, $m = 2, \cdots, n$. The mth column multiplied by $\lambda(m)/\lambda(1)$ add to the first column, $m = 2, 3, \cdots, n$. We obtain the following matrix:

$$B = \begin{pmatrix} \frac{\langle \lambda(k)\varphi(k)\rangle \mathrm{e}^{-\rho\tau}}{\langle k\rangle} & \dfrac{\lambda(1)\varphi(2)P(2)\mathrm{e}^{-\rho\tau}}{\langle k\rangle}) & \cdots & \dfrac{\lambda(1)\varphi(n)P(n)\mathrm{e}^{-\rho\tau}}{\langle k\rangle} \\ 0 & -(a+b) & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -(a+b) \end{pmatrix}.$$

The associated characteristic equation of system (5) is

$$|\rho E - A| = |\rho E - B| = (\rho + (a+b))^{n-1}(\rho - \frac{\langle \lambda(k)\varphi(k)\rangle \mathrm{e}^{-\rho\tau}}{\langle k\rangle} + (a+b)) = 0,$$

$\rho = 0$ and $\rho = -(a+b)$ are roots and the other roots satisfy

$$\rho = \frac{\langle \lambda(k)\varphi(k)\rangle \mathrm{e}^{-\rho\tau}}{\langle k\rangle} - (a+b). \tag{11}$$

Let

$$F(\rho) = \rho - \frac{\langle \lambda(k)\varphi(k)\rangle \mathrm{e}^{-\rho\tau}}{\langle k\rangle} + (a+b).$$

Note that $F(\rho)$ is continuous on $[0, +\infty)$, and

$$F(0) = 0, \quad F(0) = -\langle \lambda(k)\varphi(k)\rangle/\langle k\rangle + (a+b) = (a+b)(1 - R_0) < 0,$$

due to $R_0 > 1$. On the other hand, $\lim_{\rho\to+\infty} F(\rho) = +\infty$. So, there at least exists a positive real constant ρ_0 such that $F(\rho_0) = 0$, that is to say, the characteristic equation of system (5) has at least a positive real root. Hence the equilibrium E_0 of system (5) is unstable.

Next, we discuss the persistence of the viruses when $R_0 > 1$. Denote

$$X = \{(I_1(t), I_2(t), \cdots, I_n(t)) | 0 \le I_k \le 1, \;\; k = 1, 2, ..., n\},$$
$$X^0 = \{(I_1(t), I_2(t), \cdots, I_n(t)) | 0 < I_k < 1, \;\; k = 1, 2, ..., n\}.$$

Let $(I_1(t), I_2(t), ..., I_n(t)) = (I_1(t, \phi), I_2(t, \phi), ..., I_n(t, \phi))$ be the solution of (5) with initial function $\phi = (\phi_1, \phi_2, ..., \phi_n)$ and

$$T(t)(\phi)(\theta) = (I_1(t + \theta, \phi), I_2(t + \theta, \phi), ..., I_n(t + \theta, \phi)), \theta \in [0, \tau].$$

Obviously, X and X^0 are positively invariant sets for $T(t)$. $T(t)$ is completely continuous for $t > \tau$. Also, it follows from (0) that $I_k(t) \le 1$ for $t > 0$, which implies $T(t)$ is point dissipative. E_0 is the unique equilibrium of system () on $\partial X^0 = X/X^0$ and $\tilde{A_\partial} = \{E_0\}$, E_0 is isolated and acyclic.

Now we prove $W^s(E_0) \bigcap X^0 = \emptyset$. Suppose it is not true, then there exists a solution $(I_1(t), I_2(t), \cdots, I_n(t))$ in X^0 such that $I_k(t) \to 0$ as $t \to \infty$.

Since $R_0 > 1$, we can choose $0 < \eta < 1$ such that $\dfrac{(1 - \eta)\langle \lambda(k)\varphi(k)\rangle}{\langle k\rangle(a + b)} > 1$.

For $\eta > 0$, there exists a $T > 0$ such that

$$0 \le I_k(t) < \eta.$$

For all $t \ge T$ and $k = 1, 2, ..., n$, we have from (2) and (5) that

$$\begin{aligned} \dot{\Theta}(t) &= \sum_k \varphi(k)P(k)(\lambda(k)(1 - I_k(t))\Theta(t - \tau) - (a + b)I_k(t)) \\ &> \frac{(1 - \eta)\langle \lambda(k)\varphi\rangle}{\langle k\rangle}\Theta(t - \tau) - (a + b)\Theta(t). \end{aligned}$$

Note that $\dfrac{(1 - \eta)\langle \lambda(k)\varphi(k)\rangle}{\langle k\rangle(a + b)} > 1$, i.e., $\dfrac{(1 - \eta)\langle \lambda(k)\varphi\rangle}{\langle k\rangle} > a + b$. According to Lemma 1, we can obtain that $\lim_{t\to+\infty} \Theta(t) = +\infty$, which contradicts to the boundedness of $\Theta(t)$. This completes the proof. □

Hence, the infection is persistent according to Lemma 2, i.e., there exists a positive constant ε, such that $\lim_{t\to\infty} \inf I(t) > \varepsilon$.

The basic reproductive number for system (5) is

$$R_0 = \langle \lambda(k)\varphi(k)\rangle / ((a + b)\langle k\rangle).$$

If $R_0 < 1$, the infection will disappear due to global stability of $E_0(0, 0, \cdots, 0)$. If $R_0 > 1$, the infection will always exists, i.e., $\lim_{t\to\infty} \inf I(t) > \varepsilon > 0$.

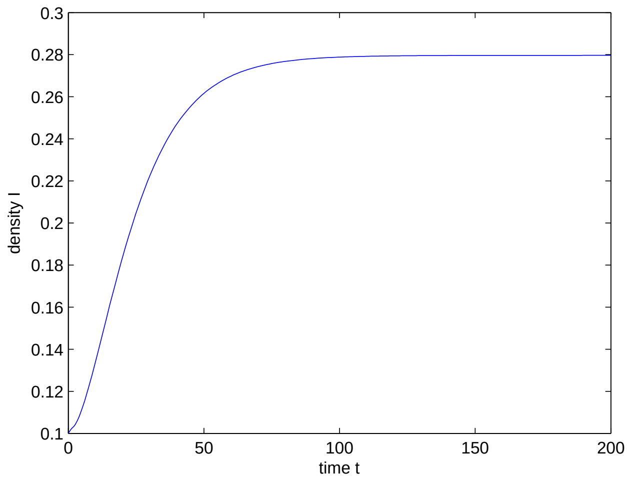

Figure 1: Evolution of infection density $I(t)$ with $R_0 = 1.67 > 1$.

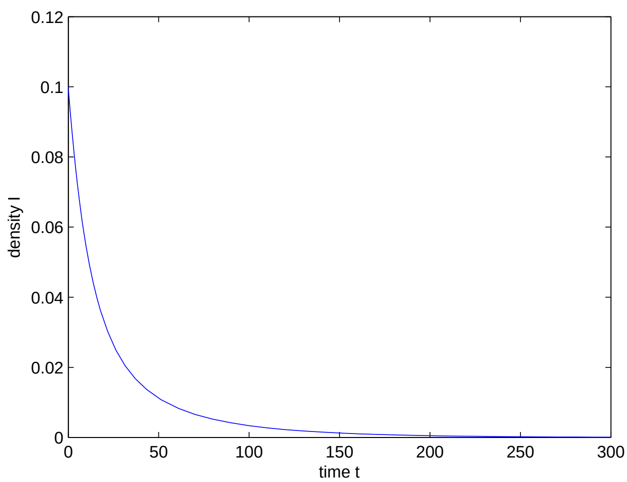

Figure 2: Evolution of infection density $I(t)$ with $R_0 = 0.835 < 1$.

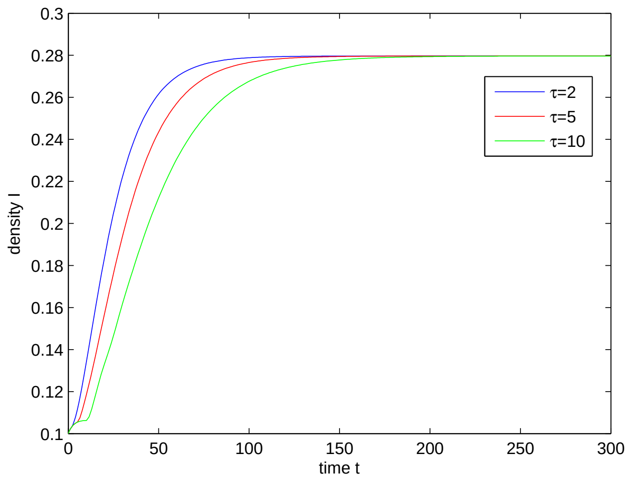

Figure 3: Evolution of infection density $I(t)$ with $\tau = 2, 5, 10$ differently when $R_0 = 1.67 > 1$.

3. Numerical Simulation

In this section, we present several numerical simulations in terms of network structure and model parameters to support the results obtained in previous sections. The simulations are based on scale-free networks, which is hererhenous, where the degree distribution $P(k) = ck^{-\gamma}$ in which constant c satisfies

$$\sum_{k=1}^{n} P(k) = 1,$$

where n is the maximum number of contact each individual.

In Fig. 1, the times series of $I(t)$ are performed with $\gamma = 2.5, a = 0.04, b = 0.06, \tau = 5, \lambda(k) = 0.1k, \varphi(k) = hk^{\alpha}/(1 + gk^{\alpha})$ in which $h = 0.5, g = 0.02, \alpha = 0.75$. In Fig. 2, the times series of $I(t)$ are performed with $\lambda(k) = 0.05k$ and other parameters have the same values as in Fig. 1. It can be seen that from Fig. 1 and Fig. 2 the diseases will eventually disappear when $R_0 < 1$ or the diseases is uniformly persistence when $R_0 > 1$. These results are in agreement with Theorem 2.

Fig. 3 shows the steady-state density of infected nodes with different values of τ when $R_0 > 1$. We show that the incubition delay has no effects on the basic reproductive number and the steady state of average density $I(t)$ of the infected codes.

4. Conclusion

In summary, to better understand the effect of the time delay on the transmission of disease, we introduce a SIS model on scale-free networks which includes the nonlinear infectivity and incubation delay. Through mathematical analysis and numerical simulations, the basic reproductive number for the model has been determined. As results indicate, the delays have no effects on the basic reproductive number and the steady state of average density. Numerical simulation demonstrate the main results.

References

[1] A.L. Barabási, R. Alber, Emergence of scaling in random networks, *Science*, **286** (1999), 509-512.

[2] R. Pastor-Satorras, A. Vespignani, Epidemic spreading in scale free networks, *Phys. Rev. Lett.*, **86** (2001), 3200-3203.

[3] R. Pastor-Satorras, A. Vespignani, Epidemic dynamics in finite size scale-free networks, *Phys. Rev. E.*, **65** (2002), Paper No 035108.

[4] H. Zhang, X. Fu, Spreading of epidemics on scale-free networks with nonlinear infectivity, *Nonlinear Analysis*, **70** (2009), 3273-3278.

[5] R. Yang, J. Ren et al., Epidemic spreading on hererogeneous networks with identical infectivity, *Phys. Lett. A*, **364** (2007), 189-193.

[6] R. Olinky, L. Stone, Unexpected epidemic thresholds in heterogeneous networks: the role of disease transmission, *Phys. Rev. E*, **70** (2004), 030902.

[7] S. Eubank, H. Guclu, A. Kumar et al., Modelling disease outbreaks in realistic urban social networks, *Nature*, **429** (2004), 180-184.

[8] M. Small, C.K. Tse, Small world and scale free model of transmission of SARS, *Int. J. Bifurcat. Chaos*, **15** (2005), 1745-1755.

[9] M. Small, D.M. Walker, C.K. Tse, Scale free distribution of avian influenza outbreaks, *Phys. Rev. Lett.*, **99** (2007), 188702.

[10] X. Cheng, X. liu, Z. Chen, Z. Yuan, Spreading behavior of SIS model with non-uniform transmission on scale-free networks, *The Journal of Universities of Posts and Telecommunications,* **16** (2009), 27-31.

[11] K. Li, M. Small, H. Zhang, X. Fu, Epidemic outbreaks on networks with effective contacts, *Nonlinear Analysis: Real World Applications*, **11** (2010), 1017-1025.

[12] Z. Ma, J. Li, *Dynamical Modelling and Analysis of Epidemics*, World Scientific Publ., Singapore, 2009.

[13] C. Xia, Z. Wang, J. Sanz, S. Meloni, Y. Moreno, Effects of delayed recovery and nonuniform transmission on the spreading of diseases in complex networks, *Physica A*, **392** (2013), 1577-1585.

[14] Q. Liu, C. Deng, M. Sun, The analysis of an epidemic model with time delay on scale-free networks, *Physica A*, **410** (2014), 79-87.

[15] J. Wang, J. Wang, M. Liu, Y. Liu, The analysis of an epidemic model with time delay on scale-free networks, *Physica A*, **410** (2014), 268-275.

[16] G. Zhu, X. Fu, G. Chen, Global attractivity of a network-based epidemics SIS model with nonlinearn infectivity, *Commun. Nonlinear Sci. Numer. Simulat.*, **17** (2012), 2588-2594.

[17] H.L. Smith, Monotone Dynamical Systems, *An Introduction to the Theory of Competitive and Cooperative Systems*, Math. Surveys and Monographs, 41, American Mathematical Society, Providence: Rhode Island, 1995.

[18] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Boston, 1993.