International Journal of Applied Mathematics

Volume 28 No. 5 2015, 557-565
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v28i5.8

A NOTE ON THE EXISTENCE OF SOLITARY WAVES TO THE REGULARIZED BENJAMIN-ONO ZAKHAROV-KUZNETSOV (rBO-ZK) EQUATION

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Abstract: In this paper, we examine the existence of solitary waves to the following equation

$$u_t + a(u^n)_x + (b\mathscr{H}u_t + u_{yy})_x = 0,$$

where $\mathscr{H}$ is the Hilbert transform with respect to x, and a and b are real numbers, with $b > 0$, via a variant of the mountain pass lemma.

AMS Subject Classification: 35A01, 35G25, 42B35, 35C08
Key Words: regularized Benjamin-Ono Zakharov-Kuznetsov equation, solitary waves, mountain pass lemma

1. Introduction

In this paper we shall present an alternative proof to that presented in [4] of the existence of solitary waves solutions to the following equation

$$u_t + a(u^n)_x + (b\mathscr{H}u_t + u_{yy})_x = 0, \tag{1.1}$$

where $\mathscr{H}$ is the Hilbert transform with respect to x, defined by

Received: July 4, 2015
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$$\mathscr{H}(f)(x,y) = \text{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi,y)}{x-\xi}\,d\xi,$$

when $f \in \mathcal{S}$, and a and b are real numbers, with $b > 0$.

This equation is a bidimensional version of the regularized Benjamin-Ono equation

$$u_t + a(u^n)_x + b\mathscr{H}u_{xt} = 0. \tag{1.2}$$

For equation (1.1) has been shown the local well-posedness in Sobolev spaces and the local well and ill-posedness in weighted Sobolev spaces, also it has been proved a property of unique continuation that implies the no persistence of solutions of this in spaces of functions with arbitrary decay polynomial (see [3]). In [5] it is proved that when considering Sobolev spaces with negative indices, the map data-solution for the equation (1.1) flow is not C_2 and therefore Picard' s iteration fails for those rough Sobolev spaces. Also, there is proved a global well-posedness result to this equation for small data and an interesting scattering property of these global solutions.

2. Preliminaries

The proof of the existence of solitary waves solutions to (1.1) presented here uses a variant of mountain pass lemma. In this section we provide some preliminary results that we shall use later. Let us recall two important lemmas whose proofs can be found in [1].

Lemma 2.1. *Let $s \in (0, n/2)$ and $f \in H^s(\mathbb{R}^n)$. Then, for p such that $s = n(\frac{1}{2} - \frac{1}{p})$, $f \in L^p(\mathbb{R}^n)$ and*

$$\|f\|_{L^p} \le c_{n,s}\|D^s f\|_{L^2} \le c_{n,s}\|f\|_s,$$

where $D^s f = (-\Delta)^{\frac{s}{2}} = (|\xi|^s \widehat{f})^{\vee}$.

Lemma 2.2. *Let s_1 and s_2 be real numbers such that $s_1 < s_2$. Suppose f is a tempered distribution such that $D^{s_1} f \in L^2$ and $D^{s_2} f \in L^2$. Then, for $s \in [s_1, s_2]$, $D^s f \in L^2$ and*

$$\|D^s f\|_{L^2} \le C_s \|D^{s_1} f\|_{L^2}^{\theta} \|D^{s_2} f\|_{L^2}^{1-\theta},$$

where

$$\theta = \frac{s_2 - s}{s_2 - s_1}.$$

Lemma 2.3. *If $f \in H^1(\mathbb{R})$, then*

$$\sup_{x\in\mathbb{R}} |f(x)| \leq \|f\|_{L^2(\mathbb{R})}^{1/2} \|f_x\|_{L^2(\mathbb{R})}^{1/2}.$$

Proof. Let $f \in H^1(\mathbb{R})$, by the fundamental theorem of calculus and the Cauchy Schwartz inequality, we have

$$f^2(x) = \int_{-\infty}^{x} 2f(z)f_x(z)\, dz \leq C\|f\|_{L^2}\|f_x\|_{L^2},$$

hence

$$\sup_{x\in\mathbb{R}} |f(x)| \leq \|f\|_{L^2(\mathbb{R})}^{1/2} \|f_x\|_{L^2(\mathbb{R})}^{1/2}.$$

□

Definition 2.1. Let

$$\mathscr{X} = \mathscr{X}(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) \mid D_x^{1/2} f \in L^2(\mathbb{R}^2) \quad and \quad \partial_y f \in L^2(\mathbb{R}^2)\} \tag{2.1}$$

be the normed space with the norm defined by

$$\|f\|_{\mathscr{X}}^2 = \|f\|_{L^2(\mathbb{R}^2)}^2 + \|D_x^{1/2} f\|_{L^2(\mathbb{R}^2)}^2 + \|\partial_y f\|_{L^2(\mathbb{R}^2)}^2. \tag{2.2}$$

It is clear that $\mathscr{X}$ is a Hilbert space with this norm.

As consequence of these three lemmas we have the following embedding lemma.

Proposition 2.1. *For $0 \leq p \leq 4$, there exists a constant C, that only depend the p, such that, for all $f \in \mathscr{X}$,*

$$\|f\|_{L^{p+2}}^{p+2} \leq C\|f\|_{L^2}^{\frac{4-p}{2}} \|D_x^{1/2} f\|_{L^2}^{p} \|\partial_y f\|_{L^2}^{\frac{p}{2}}.$$

In particular, if $f \in \mathscr{X}$

$$\|f\|_{L^{p+2}} \leq C\|f\|_{\mathscr{X}}.$$

Proof. First suppose that $p < 4$. By Lemma 2.3, the H older inequality and the Minkowski integral inequality, we have that

$$\begin{aligned}\int_{\mathbb{R}^2} |f(x,y)|^{p+2}\, dxdy &\leq \int_{-\infty}^{\infty} \sup_{y\in\mathbb{R}} |f(x,y)|^p \int_{-\infty}^{\infty} f(x,y)^2\, dy\, dx \\ &\leq C\int_{-\infty}^{\infty} \|f(x,\cdot)\|_{L^2(\mathbb{R})}^{p/2} \|\partial_y f(x,\cdot)\|_{L^2(\mathbb{R})}^{p/2} \|f(x,\cdot)\|_{L^2(\mathbb{R})}^2\, dx\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x,y)^2\,dy\right)^{\frac{p+4}{4}}\left(\int_{-\infty}^{\infty}(\partial_y f(x,y))^2\,dy\right)^{\frac{p}{4}}dx\\
&\leq C\|\partial_y f\|_{L^2}^{\frac{p}{2}}\left[\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x,y)^2\,dy\right)^{\frac{p+4}{4-p}}dx\right]^{\frac{4-p}{4}}\\
&\leq C\|\partial_y f\|_{L^2}^{\frac{p}{2}}\left[\int_{-\infty}^{\infty}\|f(\cdot,y)\|^2_{L^{\frac{2(p+4)}{4-p}}}\,dy\right]^{\frac{p+4}{4}}.
\end{aligned}
\tag{2.3}
$$

Now, by Lemma 2.1,

$$
\|f(\cdot,y)\|^2_{L^{\frac{2(p+4)}{4-p}}}\leq C\|D_x^{\frac{p}{p+4}}f(\cdot,y)\|^2_{L^2}. \tag{2.4}
$$

On the other hand, Lemma 2.2,

$$
\|D_x^{\frac{p}{p+4}}f(\cdot,y)\|^2_{L^2(\mathbb{R})}\leq C\|f(\cdot,y)\|_{L^2(\mathbb{R})}^{\frac{2(4-p)}{p+4}}\|D_x^{1/2}f(\cdot,y)\|_{L^2(\mathbb{R})}^{\frac{4p}{p+4}}. \tag{2.5}
$$

Then, the (2.3), (2.4), (2.5) and the H older inequality, we have

$$
\|f\|^{p+2}_{L^{p+2}}\leq C\|\partial_y f\|_{L^2}^{\frac{p}{2}}\|f\|_{L^2}^{\frac{4-p}{2}}\|D_x^{1/2}f\|^p_{L^2}. \tag{2.6}
$$

Now, we show the case $p=4$. By Lemma 2.1, for all $u\in H^1(\mathbb{R})$ we have that

$$
\|u\|_{L^6}\leq C\|D^{\frac{1}{12}}u\|_{L^4}\leq\|u\|_{L^4}^{\frac{2}{3}}\|D^{\frac{1}{4}}u\|_{L^4}^{\frac{1}{3}}\leq\|u\|_{L^4}^{\frac{2}{3}}\|D^{\frac{1}{2}}u\|_{L^2}^{\frac{1}{3}},
$$

then, for all $f\in\mathcal{S}(\mathbb{R}^2)$,

$$
\int_{\mathbb{R}^2} f^6(x,y)\,dxdy\leq C\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f^4(x,y)\,dx\right)\left(\int_{-\infty}^{\infty}(D_x^{1/2}f)^2(x,y)\,dx\right)dy.
$$

On the other hand,

$$
\begin{aligned}
f^4(x,y)=&4\int_{-\infty}^{y} f^3(x,\tilde{y})f_y(x,\tilde{y})\,d\tilde{y}\\
\leq&4\left(\int_{-\infty}^{\infty} f^6(x,\tilde{y})\,d\tilde{y}\right)^{1/2}\left(\int_{-\infty}^{\infty} f_y^2(x,\tilde{y})\,d\tilde{y}\right)^{1/2},
\end{aligned}
$$

we have

$$
\int_{-\infty}^{\infty} f^4(x,y)\,dx\leq 4\left(\int_{\mathbb{R}^2} f^6(x,\tilde{y})\,dxd\tilde{y}\right)^{1/2}\left(\int_{\mathbb{R}^2} f_y^2(x,\tilde{y})\,dxd\tilde{y}\right)^{1/2}.
$$

so,

$$\int_{\mathbb{R}^2} f^6(x,y)\,dxdy \leq C\left(\int_{\mathbb{R}^2} f^6(x,y)\,dxdy\right)^{1/2} \|D_x^{1/2} f\|^2 \|f_y\|,$$

it follows that

$$\int_{\mathbb{R}^2} f^6(x,y)\,dxdy \leq C\|D_x^{1/2} f\|^4 \|f_y\|^2.$$

This shows this proposition. □

The following two lemmas are similar to Lemmas 2.11 and 2.12 in [2] and their proofs follow the same ideas.

Lemma 2.4. *For $0 \leq p < 4$ the embedding $\mathscr{X} \hookrightarrow L^p_{loc}(\mathbb{R}^2)$ is compact. In other words, if $\{\phi_n\}$ is a bounded sequence in $\mathscr{X}$ and $R > 0$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which converges strongly to u in $L^p(B_R)$.*

Lemma 2.5. *If $\{u_n\}$ is bounded in $\mathscr{X}$ and*

$$\lim_{n\to\infty} \sup_{(x,y)\in\mathbb{R}^2} \int_{B(x,y,R)} |u_n|^2\,dxdy = 0. \tag{2.7}$$

then $u_n \to 0$ in $L^p(\mathbb{R}^2)$.

Definition 2.2. Suppose E is a real Banach space and $I \in C^1(E,\mathbb{R})$. I satisfies the Palais-Smale condition at level c, if there exists a sequence $\{u_m\}$ in E, such that $I(u_m) \to c$ and $\lim\limits_{m\to\infty} I'(u_m) = 0$.

Lemma 2.6. *Suppose E is a Banach space and $I \in C^1(E,\mathbb{R})$ satisfies the following properties:*

1. *$I(0) = 0$, and there exist $\rho > 0$ and $\alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha > 0$.*
2. *There exist $\beta \in E - \overline{B_\rho(0)}$ such that $I(\beta) \leq 0$.*

Let Γ be the set all paths which connects 0 and β, i.e.,

$$\Gamma = \{g \in C([0,1],E) \mid g(0) = 0, g(1) = \beta\}$$

and

$$c = \inf_{g\in\Gamma} \max_{t\in[0,1]} I(g(t)).$$

Then $c \geq \alpha$ and I possesses a Palais-Smale sequence at level c.

Proof. See Theorem 2.8 in [6]. □

3. Existence of Solitary Waves

If $\phi(x-ct,y)$ is a solitary wave solution to (1.1), then

$$-c\phi_x + a(\phi^n)_x + (-cb\mathcal{H}\phi_x + \phi_{yy})_x = 0. \tag{3.1}$$

If $\phi \in \mathscr{X}$, we can write (3.1) as

$$-c\phi + a\phi_n - cb\mathcal{H}\phi_x + \phi_{yy} = 0. \tag{3.2}$$

Then ϕ is a critical point of the functional I on $\mathscr{X}$ defined as

$$I(\phi) = \int_{\mathbb{R}^2} \frac{1}{2}\left[c\phi^2 - a\frac{\phi^{n+1}}{n+1} + cb(D_x^{1/2}\phi)^2 + (\partial_y\phi)^2\right] dxdy. \tag{3.3}$$

Therefore, in order to ensure the existence of solitary waves solutions to the equation (1.1) it is enough prove that I have non-zero critical points in $\mathscr{X}$.

Let us see that I satisfies the conditions or Lemma 2.6. It is obvious that I is a C^1 functional for $0 < p \leq 4$ and $I(0) = 0$. Let $\psi \in \mathscr{X}$ be such that $\|\psi\|_{\mathscr{X}} = 1$. Then for $\alpha \in \mathbb{R}$ we have

$$\begin{aligned}
I(\alpha\psi) &= \int_{\mathbb{R}^2} \frac{1}{2}\left[c(\alpha\phi)^2 - a\frac{(\alpha\psi)^{n+1}}{n+1} + cb(D_x^{1/2}(\alpha\psi))^2 + (\partial_y(\alpha\psi))^2\right] dxdy \\
&\geq \frac{\min\{c, cb, 1\}}{2}\alpha^2\left[\int_{\mathbb{R}^2} \psi^2 + (D_x^{1/2}\psi)^2 + (\partial_y\psi)^2\, dxdy\right] \\
&\quad - a\frac{\alpha^{n+1}}{n+1}\int_{\mathbb{R}^2} \psi^{n+1}\, dxdy \\
&\geq \frac{\min\{c, cb, 1\}}{2}\alpha^2\|\psi\|_X^2 - a\frac{\alpha^{n+1}}{n+1}\|\psi\|_{L^{p+1}(\mathbb{R}^2)}^{p+1} \\
&\geq \frac{\min\{c, cb, 1\}}{2}\alpha^2 - a\frac{\alpha^{n+1}}{n+1} \\
&= \alpha^2\left[\frac{\min\{c, cb, 1\}}{2} - a\frac{\alpha^{n-1}}{n+1}\right].
\end{aligned}$$

Then, taking α' small enough, for instance

$$\alpha' < \sqrt[n+1]{\frac{(n+1)\min\{c, cb, 1\}}{2a}},$$

we have that $I|_{\partial B_{\alpha'}(0)} \geq \rho > 0$, where $\rho = \left[\frac{\min\{c,cb,1\}}{2}\alpha^2 - a\frac{\alpha^{n+1}}{n+1}\right]$.

Let $\psi \in \mathscr{X}$ fixed such that $\|\psi\|_{\mathscr{X}} = 1$ y $\|\psi\|_{L^{p+1}}^{p+1} = c$, we have that

$$I(\alpha\psi) = \frac{1}{2}K\alpha^2 - \frac{1}{n+1}L\alpha^{n+1}.$$

Taking α small enough, we have, $I(\alpha\psi) < 0$. Also α can be taken large enough such that $e := \alpha\psi \in E - \overline{B_{\alpha'}(0)}$. This prove the second condition of Lemma 2.6. So, we have shown the following lemma.

Lemma 3.1. *Let I, α y β be defined as above and let Γ and c be defined as in Lemma 2.6. Then, there exists a sequence $\{\phi_n\}$, such that $I(\phi_n) \to c$ and $I'(\phi_n) \to 0$.*

Now, we can prove the following theorem.

Theorem 3.1. *The equation* (3.1) *has nontrivial solutions in $\mathscr{X}$.*

Proof. It is enough to show that I have non-zero critical points in $\mathscr{X}$. By Lemma 3.1, there exists a Palais-Smale sequence $\{\phi_n\}$ at level c of I. Therefore,

$$\begin{aligned}
c + o(1)\|\phi_n\|_{\mathscr{X}} &\geq I(\phi_n) - \frac{(I'(\phi_n), \phi_n)_{\mathscr{X}}}{p+1} \\
&= \int_{\mathbb{R}^2} \frac{1}{2}(c\phi_n^2 + bc(D_x^{1/2}\phi_n)^2 + (\partial_y\phi_n)^2) - a\frac{\phi_n^{p+1}}{p+1}\,dxdy- \\
&- \frac{1}{p+1}\int_{\mathbb{R}^2} (c\phi_n^2 + cb(D_x^{1/2}\phi_n)^2 + (\partial_y\phi_n)^2) - a\phi_n^{p+1}\,dxdy \\
&\geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\min\{c, cb, 1\}\|\phi_n\|_{\mathscr{X}}^2,
\end{aligned}$$

for n big enough. Hence $\{\phi_n\}$ is bounded in $\mathscr{X}$. Considering that

$$\begin{aligned}
0 < c &= \lim_{n\to\infty} I(\phi_n) - \frac{1}{2}(I'(\phi_n), \phi_n)_{\mathscr{X}} \\
&= \lim_{n\to\infty}\left(\int_{\mathbb{R}^2} \frac{1}{2}(c\phi_n^2 + bc(D_x^{1/2}\phi_n)^2 + (\partial_y\phi_n)^2) - a\frac{\phi_n^{p+1}}{p+1}\,dxdy-\right. \\
&\left. - \frac{1}{2}\int_{\mathbb{R}^2} (c\phi_n^2 + cb(D_x^{1/2}\phi_n)^2 + (\partial_y\phi_n)^2) - a\phi_n^{p+1}\,dxdy\right) \\
&= \lim_{n\to\infty} \frac{a(p-1)}{2(p+1)}\int_{\mathbb{R}^2} \phi_n^{p+2}\,dxdy.
\end{aligned}$$

Lemma 2.5 implies that

$$\delta = \limsup_{n\to\infty} \sup_{(x,y)\in\mathbb{R}^2} \int\limits_{(x,y)+\Omega} \phi_n^2 \, dxdy > 0,$$

Then, passing to a subsequence if necessary, we can assume that there exists a sequence $(x_n, y_n) \in \mathbb{R}^2$ such that

$$\int\limits_{(x,y)+\Omega} \phi_n^2 \, dxdy > \frac{\delta}{2} \tag{3.4}$$

for n big enough. Let $\widetilde{\phi}_n = \phi_n(\cdot - (x_n, y_n))$. Then, again passing to a subsequence if necessary, we can assume that, for some $\phi \in \mathscr{X}$, $\widetilde{\phi}_n \longrightarrow \phi$ in $\mathscr{X}$. In view of (3.4), for n large enough, and Lemma 2.4, $\phi \neq 0$. Lemma 2.4 and the continuity of the function $u \to u^{p+1}$, imply that

$$I'(\phi)(\omega) = \lim_{n\to\infty} I'(\phi_n)(\omega) = 0,$$

for all $\omega \in \mathscr{X}$. This shows this theorem. □

Acknowledgments

The authors were supported by the Universidad Central.

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