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POSITIVE SOLUTIONS FOR SOME NONLINEAR DELAY INTEGRAL EQUATIONS

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Abstract: In this work we give an extension of the results obtained in [2], we are interested in producing sufficient conditions for the existence of positive periodic solution to

$$x(t) = \int_0^{\tau(t)} h(t, s, x(t - s - l))\, ds,$$

in the special case where $h(t, s, x) = f(t, s, x)g(t, s, x)$.

For it, we use topological methods, more precisely, the fixed point index.

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1. Introduction

In their paper [2], Cañada and Zertiti have studied integral equations of type

$$x(t) = \int_0^{\tau(t)} h(t, s, x(t - s - l))\, ds, \tag{1}$$

which formulate a model to explain the evolution of certain infectious diseases and it may also be considered as a growth equation for single species populations when the birth rate varies seasonally. It includes, as a particular case, different

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equations suggested by other authors (see [2, 4, 7, 12, 11]). For instance, Cooke and Kaplan in [4] have defined a number β (dependent on f), such that for each $\tau > \beta$ the equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\, ds$$

has a positive solution of period ω. Also, in [12] Torrejon has shown the existence of nontrivial almost periodic solution of the equation

$$x(t) = \int_{t-\sigma(x(t))}^{t} f(s, x(s))\, ds,$$

where f is a continuous nonnegative almost periodic function in t. And in [11], Nussbaum has shown that nontrivial ω-periodic solutions of

$$x(t) = \int_{t-\tau}^{t} P(t-s, \tau) f(s, x(s))\, ds,$$

bifurcate from the trivial solution ($f(t, 0) \equiv 0$) when τ exceeds a certain threshold value τ_0. Finally in [2], Cañada and Zertiti have given a condition (sufficient and necessary) for the existence of solution to (1).

In this work we give an extension of results obtained in [2]. We are interested in producing sufficient conditions for the existence of positive periodic solution to (1) in the special case where $h(t, s, y) = f(t, s, y)g(t, s, y)$ under the following assumptions on functions f and g :
$f, g : \mathbb{R} \times \mathbb{R} \times [0, +\infty[\longrightarrow \mathbb{R}$ are continuous functions with:

(F1) : $f(t, s, 0) = 0$ for all $(t, s) \in \mathbb{R} \times \mathbb{R}$,

(F2) : $f(t, s, y) \geq 0, g(t, s, y) \geq 0, \forall (t, s, y) \in \mathbb{R} \times \mathbb{R} \times [0, +\infty[$ and there exists a positive number $w, (w > 0)$ such that $f(t + w, s, y) = f(t, s, y)$ and $g(t + w, s, y) = g(t, s, y), \forall (t, s, y) \in \mathbb{R} \times \mathbb{R} \times [0, +\infty[$,

(F3) : l is a nonnegative constant and $\tau : \mathbb{R} \longrightarrow \mathbb{R}^+$ is a continuous and λ-periodic function ($\lambda > 0$) such that $\frac{\omega}{\lambda} = \frac{p}{q}, p, q \in \mathbb{N}$.

In [4, 11], the authors assume that $\lim_{y \to +\infty} \frac{f(t,s,y)}{y} = 0$ uniformly in (t, s), and in [2], they assume that there exists a continuous function $b : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $\lim_{y \to +\infty} \frac{f(t,s,y)}{y} = b(t, s)$ uniformly in (t, s). In this paper we allow f to have more general asymptotic behavior at infinity (see Theorems 1 and 2 below). In Theorem 2 we suppose that $g(t, s, y) \equiv 1$ and there exist a continuous function $b : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $p \geq 1$ such that $\lim_{y \to +\infty} \frac{f(t,s,y)^p}{y} = b(t, s)$ uniformly in (t, s). Then we have two cases: $p = 1$ and $p > 1$. If $p = 1$ we refer to results of [2] for more details, and if $p > 1$ we apply Theorem 1 below, which extends the results of [2].

We finish this paper by proving several results about multiplicity of solutions, in the line of Guo and Lakshmikantham [7]. Some examples are given to illustrate our results.

2. Main Results

Denote by P the cone of nonnegative functions in the real Banach space E, of all real and continuous $q\omega-$ periodic functions defined on $\mathbb{R}$, where if $x \in E$

$$\|x\| = \max_{0\leq t\leq q\omega} |x(t)|.$$

Let us consider the integral equation

$$x(t) = \int_0^{\tau(t)} f(t,s,x(t-s-l))g(t,s,x(t-s-l))\, ds. \tag{2}$$

In this section we are interested in the existence of solution of (2) in $P \setminus \{0\}$. Define the operator $F : E \longrightarrow E$ by

$$Fx(t) = \int_0^{\tau(t)} f(t,s,x(t-s-l))g(t,s,x(t-s-l))\, ds.$$

Then equation (2) has a continuous, nonnegative, and nontrivial $q\omega$-periodic solution iff there exists $x \in P \setminus \{0\}$ verifying

$$x = Fx.$$

Now, we present and prove our main results.

Theorem 1. *Suppose that :*

(H1) there exists a continuous function $a : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\lim_{y\to 0^+} \frac{f(t,s,y)g(t,s,y)}{y} = a(t,s), \quad \textit{uniformly in } (t,s) \in \mathbb{R}\times\mathbb{R},$$

(H2) there exists a continuous function $b : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $p > 1$ such that

$$\lim_{y\to +\infty} \frac{f(t,s,y)^p}{y} = b(t,s), \quad \textit{uniformly in } (t,s) \in \mathbb{R}\times\mathbb{R},$$

(H3) there exists $\mu > 0$ such that for all $x \in P$, we have

$$\int_0^{\tau(t)} g(t,s,x(t-s-l))^q ds < \mu, \quad \forall t \in \mathbb{R},$$

where q verifying $\frac{1}{p} + \frac{1}{q} = 1$

(H4) $\overset{\circ}{A}_t = \emptyset \quad \forall t \in \mathbb{R}$, *where* $A_t = \{s \in \mathbb{R} : a(t, t-s) = 0\}$.

Then if

$$r(L(\tau, a)) > 1, \quad \text{and} \quad r(L(\tau, b)) < p, \tag{3}$$

equation (2) *has a solution in* $P \setminus \{0\}$, *where* $r(L(\tau, a))$ *means the spectral radius of the linear operator* $L(\tau, a) : E \longrightarrow E$ *defined by*

$$L(\tau, a)x(t) = \int_0^{\tau(t)} a(t, s)x(t-s-l)\, ds, \quad \forall x \in E,$$

(analogously for $r(L(\tau, b))$ *and* $L(\tau, b)$*).*

Proof. We must observe that (E, P) is an ordered Banach space with $\overset{\circ}{P} \neq \emptyset$. Also it is not difficult to see that $F : P \to P$ is completely continuous. Moreover:

(a) We claim that there exists $R > 0$ such that if $x \in P$ and $t \in [0, 1]$ satisfy $x = tFx$, then $\|x\| \leq R$.

Suppose that is not true, then we can find sequences $\lambda_n \in [0, 1]$ and $x_n \in P$ such that $\|x_n\| \longrightarrow \infty$ and

$$x_n = \lambda_n F x_n \leq F x_n,$$

where the partial ordering is that induced by the cone P ($x, y \in E$ then $x \leq y$ if $y - x \in P$).

By using the young inequality we get

$$\begin{aligned} Fx_n(t) &= \int_0^{\tau(t)} f(t, s, x_n(t-s-l))g(t, s, x_n(t-s-l))ds \\ &\leq \frac{1}{p}\int_0^{\tau(t)} f(t, s, x_n(t-s-l))^p\, ds + \\ &+ \frac{1}{q}\int_0^{\tau(t)} g(t, s, x_n(t-s-l))^q\, ds \\ &= \frac{1}{p}G_1x_n(t) + \frac{1}{q}G_2x_n(t), \end{aligned}$$

where

$$G_1x_n(t) = \int_0^{\tau(t)} f(t, s, x_n(t-s-l))^p\, ds$$

and

$$G_2x_n(t) = \int_0^{\tau(t)} g(t, s, x_n(t-s-l))^q\, ds,$$

so that

$$\frac{x_n}{\|x_n\|} \leq \frac{1}{p}\frac{G_1 x_n}{\|x_n\|} + \frac{1}{q}\frac{G_2 x_n}{\|x_n\|}$$
$$= \frac{1}{p}\frac{G_1 x_n - L(\tau,b)x_n}{\|x_n\|} + \frac{1}{p}\frac{L(\tau,b)x_n}{\|x_n\|} + \frac{1}{q}\frac{G_2 x_n}{\|x_n\|}.$$

By letting $\delta_n = \frac{x_n}{\|x_n\|}$, we get

$$\frac{1}{p}\frac{G_1 x_n - L(\tau,b)x_n}{\|x_n\|} + \frac{1}{p}L(\tau,b)\delta_n + \frac{1}{q}\frac{G_2 x_n}{\|x_n\|} - \delta_n \in P. \tag{4}$$

By using the fact that $L(\tau,b)$ is a positive linear map, we obtain

$$L(\tau,b)\Big(\frac{1}{p}\frac{G_1 x_n - L(\tau,b)x_n}{\|x_n\|}\Big) + L(\tau,b)\Big(\frac{1}{p}L(\tau,b)\delta_n\Big)$$
$$+L(\tau,b)\Big(\frac{1}{q}\frac{G_2 x_n}{\|x_n\|}\Big) - L(\tau,b)\delta_n \in P. \tag{5}$$

It is not difficult to see that [2, Theorem 2.1],

$$\lim_{\substack{\|x_n\|\longrightarrow\infty \\ x_n\in P}} \frac{1}{p}\frac{G_1 x_n - L(\tau,b)x_n}{\|x_n\|} = 0,$$

And by virtue of (H3), we easily get $\frac{1}{q}\frac{G_2 x_n}{\|x_n\|} \longrightarrow 0$ as $n \longrightarrow \infty$.

On the other hand, since $L(\tau,b)$ is compact, we may as well assume that $L(\tau,b)\delta_n \to y \in P$. Then by letting $n \to \infty$ in (5) we get

$$\frac{1}{p}L(\tau,b)y - y \in P.$$

An easy induction argument shows that

$$\frac{1}{p^n}L^n(\tau,b)y - y \in P,$$

for every positive integer n. Since $r(\frac{1}{p}L(\tau,b)) < 1$ it follows that $\frac{1}{p^n}L^n(\tau,b)y \to 0$ and then $-y \in P$, which implies that $y = 0$ since $y \in P$.

On the other hand from (4) we have

$$0 \leq \delta_n \leq \frac{1}{p}\frac{G_1 x_n - L(\tau,b)x_n}{\|x_n\|} + \frac{1}{p}L(\tau,b)\delta_n + \frac{1}{q}\frac{G_2 x_n}{\|x_n\|}.$$

Then, we can easily get

$$1 \leq \|\frac{1}{p}\frac{G_1 x_n - L(\tau,b)x_n}{\|x_n\|} + \frac{1}{p}L(\tau,b)\delta_n + \frac{1}{q}\frac{G_2 x_n}{\|x_n\|}\|.$$

Since we have already shown that

$$\frac{1}{p}\frac{G_1x_n - L(\tau,b)x_n}{\|x_n\|} + \frac{1}{p}L(\tau,b)\delta_n + \frac{1}{q}\frac{G_2x_n}{\|x_n\|} \to 0 \quad \text{as} \quad n \to \infty$$

we have arrived at a contradiction.

(b) We shall prove the existence of $F'_+(0)$ the right derivative of F along P at 0.

For this, we must prove that

$$F'_+(0)(x)(t) = L(\tau,a)x(t), \quad \forall x \in E,$$

or what is the same

$$\lim_{\substack{x \longrightarrow 0 \\ x\in P}} \frac{Fx - L(\tau,a)x}{\|x\|} = 0,$$

or equivalently

$$\forall \varepsilon \in \mathbb{R}^+, \exists r(\varepsilon) \in \mathbb{R}^+ : \|x\| \leq r(\varepsilon)(x \in P) \Rightarrow \frac{\|Fx - L(\tau,a)x\|}{\|x\|} \leq \varepsilon.$$

Let $\varepsilon > 0$. Then from (H1) there is $r(\varepsilon) \in \mathbb{R}^+$ such that

$$\mid f(t,s,y)g(t,s,y) - a(t,s)y \mid \leq \varepsilon \mid y \mid$$

$$\forall (t,s) \in \mathbb{R}\times\mathbb{R}, \forall y \in \mathbb{R} : 0 \leq y \leq r(\varepsilon).$$

Then if $x \in P$ satisfies $\|x\| \leq r(\varepsilon)$, we obtain

$$\mid Fx(t) - L(\tau,a)x(t) \mid$$

$$\begin{aligned} &\leq \int_0^{\tau(t)} \mid f(t,s,x(t-s-l))g(t,s,x(t-s-l)) \\ &\quad - a(t,s)x(t-s-l) \mid ds \\ &\quad \leq \int_0^{\tau(t)} \mid \varepsilon x(t-s-l) \mid ds \\ &\quad \leq \varepsilon\tau^*\|x\|, \quad \forall t \in \mathbb{R}. \quad (\tau^* = \max_{0\leq t\leq\lambda} \tau(t)). \end{aligned}$$

Consequently,

$$\|Fx - L(\tau,a)x\| \leq \varepsilon\tau^*\|x\|, \quad \forall x \in P : \|x\| \leq r(\varepsilon).$$

(c) It is easily seen (see [2, Theorem 2.1]) that $F'_+(0)$ is strongly positive.

Now, from (a), we obtain that $i(F,P_R) = 1$. And from (b), (c) and (3), we obtain by using Lemma 13.1 in Amann [1], that there exists $r < R$, $r > 0$ such that $i(F,P_r) = 0$ (where, for any $\rho > 0, P_\rho = \{x \in P : \|x\| < \rho\}$ and $i(F,P_\rho)$ means the fixed point index with respect to the cone P).

Then, (by the additivity property) $i(F, P_R \setminus \overline{P_r}) = 1 \neq 0$, which implies the existence of a solution of (2) in $P\setminus\{0\}$.

Remark 1. If we take $g(t,s,y) \equiv 1$ in the above theorem, we obtain the following result which is a generalization of Theorem 2.1 in [2] .

Theorem 2. *Let us suppose the following hypotheses*

*(H*1) there exists a continuous function* $a : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$\lim_{y\to 0^+} \frac{f(t,s,y)}{y} = a(t,s), \textit{uniformly in } (t,s) \in \mathbb{R}\times\mathbb{R},$$

*(H*2) there exist a continuous function* $b : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *and* $(p \geq 1)$ *such that*

$$\lim_{y\to +\infty} \frac{f(t,s,y)^p}{y} = b(t,s), \textit{uniformly in } (t,s) \in \mathbb{R}\times\mathbb{R},$$

*(H*3)* $\overset{\circ}{A_t} = \emptyset \quad \forall t \in \mathbb{R}$, *where* $A_t = \{s \in \mathbb{R} : a(t,t-s) = 0\}$.

Then if

$$r(L(\tau,a)) > 1 \quad \textit{and} \quad r(L(\tau,b)) < p, \tag{6}$$

equation (2) *has a solution in* $P \setminus \{0\}$.

Proof. We discuss two steps:
If $p = 1$, we get Theorem 2.1 in [2].
If $p > 1$, we apply the above theorem (1) by observing that

$$\int_0^{\tau(t)} 1ds < \tau^* + 1.$$

Now we present an example of Theorem 1 which cannot be derived from the results of [2].

Example 3. Let $h : [0,+\infty] \to \mathbb{R}^+$ be a continuous function verifying

$$h(0) = 0 \quad (\sqrt{h})'(0) = \alpha > 0, \quad \lim_{y\to+\infty} \frac{h(y)}{y} = \beta > 0$$

and take $d : \mathbb{R} \to \mathbb{R}$ a continuous, positive and ω-periodic function ($\omega > 0$) and $l = 0$. If for all $(t,s,y) \in \mathbb{R}\times\mathbb{R}\times[0,+\infty]$, $f(t,s,y) = \sqrt{d(t-s)h(y)}$, and

$$g(t,s,y) = \begin{cases} 1 & 0 \leq y \leq 2\pi \\ 1 + s^2\sin^2 y & y \geq 2\pi. \end{cases}$$

Then by observing that $\displaystyle\int_0^{\tau(t)} g(t,s,x(t-s-l))^2 ds < \tau^* + \frac{2\tau^{*3}}{3} + \frac{\tau^{*5}}{5} + 1$, hypotheses (H1)-(H4) of Theorem 1 are satisfied with $p = 2, q = 2$, $a(t,s) = \alpha\sqrt{d(t-s)}$, and $b(t,s) = \beta d(t-s)$.

Consequently, if

$$1 < r(L(\tau,a)), \quad r(L(\tau,b)) < 2, \tag{7}$$

equation (2) has a solution in $P \setminus \{0\}$.

Note that in the particular case where $d(t) \equiv d \in \mathbb{R}^+$ conditions (7) are satisfied if we take

$$\frac{1}{\alpha\sqrt{d}} < \min_{t\in\mathbb{R}} \tau(t) \leq \max_{t\in\mathbb{R}} \tau(t) < \frac{2}{\beta d}.$$

Here we use that fact that

$$\min_{t\in\mathbb{R}} \int_0^{\tau(t)} a(t,s)ds \leq r(L(\tau,a)), \quad r(L(\tau,b)) \leq \max_{t\in\mathbb{R}} \int_0^{\tau(t)} b(t,s)ds,$$

(see [11]).

3. Multiplicity of Solutions

In this section, we present several results about multiplicity of solutions, which generalize some previous ones by Guo and Lakshmikantham [7], and [2]. An example is given to illustrate the obtained generalizations.

Theorem 4. *Let f and g satisfy (H1),(H2), and (H3) of Theorem* (1) *and*

(H5) $r(L(\tau,a)) < 1$ *and* $r(L(\tau,b)) < p \quad (p > 1)$,

(H6) There exist $r, R, 0 < r < R$ *such that*

$$r < \int_0^{\tau(t)} f(t,s,x(t-s-l))g(t,s,x(t-s-l))\, ds < R,$$

for all $t \in \mathbb{R}$ *and for all* $x \in P$ *satisfying* $r \leq x(s) \leq R, \quad \forall s \in \mathbb{R}$.

Then equation (2) *has at least two solutions in* $P \setminus \{0\}$.

Proof. As we have seen in the proof of the above Theorem 1 there exists $R_1 > R$ such that

$$i(F, P_\sigma) = 1, \quad \forall \sigma \geq R_1.$$

Now, the proof literally repeats the last part the proof from [2, Theorem 3.1]

Remark 2. If we take $g(t,s,y) \equiv 1$ in the above theorem, we obtain the following result which is a generalization of Theorem 3.1 in [2].

Theorem 5. *Let f satisfy (H*1),(H*2) of Theorem* (2) *and:*

*(H*4)* $r(L(\tau, a)) < 1$ *and* $r(L(\tau, b)) < p$,

*(H*5) there exist r, R, $0 < r < R$ such that*

$$r < \int_0^{\tau(t)} f(t,s,x(t-s-l))\, ds < R$$

for all $t \in \mathbb{R}$ and for all $x \in P$ satisfying $r \leq x(s) \leq R$, $\forall s \in \mathbb{R}$.

Then equation (2) *has at least two solutions in $P \setminus \{0\}$.*

Proof. We discuss two steps:
If $p = 1$ we get Theorem 3.1 in [2].
If $p > 1$, we apply Theorem (4) above.

Now we give an example of Theorem 4.

Example 6. Let $h : [0, +\infty] \to \mathbb{R}^+$ be a continuous function defined by

$$h(x) = \begin{cases} x(1-x), & 0 \leq x \leq \frac{1}{2} \\ \phi(x), & \frac{1}{2} \leq x \leq 1 \\ \sqrt{2x}, & x \geq 1 \end{cases},$$

where $\phi : [\frac{1}{2}, 1] \to [0, +\infty)$ is a continuous function with $\phi(\frac{1}{2}) = \frac{1}{4}, \phi(1) = \sqrt{2}$ and take $\alpha : \mathbb{R} \to \mathbb{R}^+$ a continuous and ω-periodic function ($\omega > 0$) satisfying

$$\frac{1}{\sqrt{2}} < \int_0^{\tau(t)} \sqrt{\alpha(t-s)}\, ds < 1 \quad \text{and} \quad \int_0^{\tau(t)} \alpha(t-s)\, ds < 1. \tag{8}$$

Let for all $(t,s,y) \in \mathbb{R} \times \mathbb{R} \times [0, +\infty]$, $f(t,s,y) = \sqrt{\alpha(t-s)}h(y)$, and

$$g(t,s,y) = \begin{cases} 1 & 0 \leq y \leq 2\pi \\ 1 + s^2 \sin^2 y & y \geq 2\pi. \end{cases}$$

By taking $\alpha^* = \max_{t\in\mathbb{R}} \alpha(t)$, if $R > 2\alpha^*\big(\tau^* + \frac{\tau^{*3}}{3}\big)^2$, then for all $x \in P$ with $1 \leq x(s) \leq R$, $\forall s \in \mathbb{R}$ we have

$$1 < \int_0^{\tau(t)} f(t,s,x(t-s-l))g(t,s,x(t-s-l))\,ds < R \quad \text{for all} \quad t \in \mathbb{R}.$$

Then, it is easily seen that hypotheses (H1)-(H3) of Theorem 4 and (H6) are satisfied with $p = 2, q = 2$, $a(t,s) = \sqrt{\alpha(t-s)}$, $b(t,s) = 2\alpha(t-s)$, $r = 1$, and $R > 2\alpha^*\big(\tau^* + \frac{\tau^{*3}}{3}\big)^2$.

From (8) we easily get

$$r(L(\tau,a)) < 1, \quad r(L(\tau,b)) < 2.$$

In the particular case where $\alpha(t) \equiv d \in \mathbb{R}^+$ $(0 < d < 2)$, these conditions are verified if

$$\frac{1}{\sqrt{2d}} < \min_{t\in\mathbb{R}} \tau(t) \leq \max_{t\in\mathbb{R}} \tau(t) < \min\{\frac{1}{d}, \frac{1}{\sqrt{d}}\}.$$

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